10. Cubature Formulas on the Sphere Invariant under Finite Groups of Rotations*

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A cubature formula on the surface of the sphere

$$(l, f) = \int_S f(\vartheta, \varphi)\, dS - \sum_{k=1}^{N} c_k f(x^{(k)}) \cong 0 \tag{1}$$

is called *invariant* under transformations of a certain group G of sphere rotations if

$$\Big(l, f\big(\vartheta_1(\vartheta, \varphi), \varphi_1(\vartheta, \varphi)\big)\Big) = \Big(l, f(\vartheta, \varphi)\Big), \tag{2}$$

where

$$\vartheta_1(\vartheta, \varphi), \quad \varphi_1(\vartheta, \varphi) \tag{3}$$

is a substitution in G.

L. A. Lyusternik and V. A. Ditkin [1, 2] have considered formulas with nodes at the vertices of an icosahedron and centers of its faces. We will show how to construct cubature formulas which are invariant under the groups of rotations of the sphere corresponding to a regular polyhedron and are valid for as many spherical harmonics as possible [3].

Theorem 1. *Let a cubature formula be invariant under G. Then it is exact for all harmonics of a given degree if and only if it is exact for all invariant harmonics $Y_n^*(\vartheta, \varphi)$ of this degree n, i.e., for those hormonics which are unchanged under rotations of the sphere belonging to G:*

$$Y_n^*\big(\vartheta_1(\vartheta, \varphi), \varphi_1(\vartheta, \varphi)\big) = Y_n^*(\vartheta, \varphi). \tag{4}$$

The proof is based on the formula

$$(l, f) = (l, f_G), \tag{5}$$

where f_G is the mean of the function f over the group[1] G:

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[1] In this paper, M is the order of G. – *Ed.*

$$f_G(x) = \frac{1}{M} \sum_{g \in G} f(gx). \tag{6}$$

Let $S(n)$ be the number of invariant harmonics of degree n. This number may be computed using the representation theory of groups, as was pointed out to the author by D. K. Faddeev.

The spherical harmonics of degree n form a $(2n+1)$-dimensional space, whose basis may be chosen to be

$$e^{im\varphi} P_n^{(|m|)}(\cos\vartheta), \quad m = 0, \pm 1, \dots, \pm n. \tag{7}$$

The group of rotations of the sphere induces the group of linear substitutions which acts on harmonics (7) and is a linear representation of the former group.

Every representation decomposes into irreducible representations on subspaces of lower dimensionality. Among them some are identity representations. The number $S(n)$ of linearly independent invariant harmonics coincides with the number of such one-dimensional identity representations included in the representation A.

The traces of the matrices of irreducible representations (the so-called characters of the representation) constitute M-dimensional vectors. It is well known that characters of distinct irreducible representations are orthogonal:

$$\sum_{k=1}^{M} \chi\big(A_k^{(j)}\big)\overline{\chi}\big(A_k^{(s)}\big) = \begin{cases} M, & A^{(j)} \sim A^{(s)}, \\ 0, & A^{(j)} \nsim A^{(s)}. \end{cases} \tag{8}$$

Obviously, all characters of the identity representations equal 1. Hence, for the number $S(n)$ we get the formula

$$S(n) = \frac{1}{M} \sum_{k=1}^{M} \chi(A_k), \tag{9}$$

where A_k are the matrices representing the group rotations.

Similar matrices have the same trace; and the rotations by the same angle about corresponding elements are similar.

There are t_1 vertices, t_2 faces, and t_3 edges in a regular polyhedron. At the vertices, q_1 of elements meet; the faces are regular q_2-gons; and the edges are the axes of rotations of order $q_3 = 2$. Obviously,

$$t_1 q_1 = t_2 q_2 = t_3 q_3 = M, \tag{10}$$

while also $\frac{1}{2}[t_1(q_1-1) + t_2(q_2-1) + t_3(q_3-1)] + 1 = M$; whence

$$t_1 + t_2 + t_3 = M + 2. \tag{11}$$