The Ubiquitous Farkas Lemma

Rakesh V. Vohra

Department of Managerial Economics and Decision Sciences
Kellogg Graduate School of Management
Northwestern University
Evanston, IL 60208
r-vohra@kellogg.northwestern.edu

Summary. Every student of linear programming is exposed to the Farkas lemma, either in its original form or as the duality theorem of linear programming. What most students don't realize is just how ubiquitous the lemma is. I've managed to make a good living by simply trying to express problems in the form of linear inequalities and then examining the Farkas alternative. It stared with my dissertation (under Saul's supervision) and continues even today. So I could think of no better gift on the occasion of Saul's birthday than a compilation of some applications of the Farkas lemma. These applications have been pulled together from a variety of different sources.

Key words: Farkas lemma, linear programming, duality.

1 Introduction

As every operations research student knows, associated with every $m \times n$ matrix A of real numbers is a problem of the following kind:

> Given $b \in \Re^m$ find an $x \in \Re^n$ such that $Ax = b$, or prove that no such x exists.

Convincing someone that $Ax = b$ has a solution (when it does) is easy. One merely exhibits the solution, and the other can verify that the solution does indeed satisfy the equations. What if the system $Ax = b$ does not admit a solution. Is there an easy way to convince another of this? Stating that one has checked all possible solutions convinces no one.

By framing the problem in the right way, one can apply the machinery of linear algebra. Specifically, given $b \in \Re^m$, the problem of finding an $x \in \Re^n$ such that $Ax = b$ can be stated as: is b in the span of the column vectors of A? This immediately yields the **Fundamental Theorem of Linear algebra** first proved by Gauss.

Theorem 1. *Let A be an $m \times n$ matrix, $b \in \Re^m$, and $F = \{x \in \Re^n : Ax = b\}$. Then* **either** $F \neq \emptyset$ **or** *there exists $y \in \Re^m$ such that $yA = 0$ and $yb \neq 0$,* **but not both**.

The proof is not hard. If $F \neq \emptyset$, we are done. So suppose that $F = \emptyset$. Then, b is not in the span of the columns of A. If we think of the span of the columns of A as a plane, then b is a vector pointing out of the plane. Thus, any vector y orthogonal to this plane (and so to every column of A) must have a non-zero dot product with b. To verify the 'not both' portion of the statement, suppose not. Then there is a x such that $Ax = b$ and a y such that $yA = 0$ and $yb \neq 0$. This implies that $yAx = yb$, a contradiction, since $yAx = 0$ and $yb \neq 0$.

The next level up in difficulty is the following question:

> Given $b \in \Re^m$, find a non-negative $x \in \Re^n$ such that $Ax = b$, or prove that no such x exists.

The requirement that x be non-negative is what causes all the difficulty. It is to Gulya Farkas we are indebted for providing us with a generalization of the fundamental theorem of Linear algebra.[1]

Theorem 2 (The Farkas Lemma). *Let A be an $m \times n$ matrix, $b \in \Re^m$, and $F = \{x \in \Re^n : Ax = b, x \geq 0\}$. Then* **either** $F \neq \emptyset$ **or** *there exists $y \in \Re^m$ such that $yA \geq 0$ and $y \cdot b < 0$* **but not both.**

I'll assume that the reader is familiar with the result and its variants.[2] Now to the applications.

2 The Duality Theorem of Linear Programming

A staple of many a homework set in a class on linear programming (LP) is to prove the Farkas lemma from LP duality or the reverse. Here is one example of how to get from the Farkas lemma to the duality theorem. I'll take the following as the form of the primal problem (P):

$$Z_P = \max\{cx : \text{ s.t. } Ax = b, x \geq 0\}.$$

The dual of course is (D):

$$Z_D = \min\{yb : yA \geq c\}.$$

[1] Gulya Farkas (1847-1930) was a Hungarian theoretical physicist. The lemma that bears his name was announced by him in 1894 and has its roots in the problem of specifying the equilibrium of a system. The associated proof was incomplete. A complete proof was published in Hungarian by Farkas in 1898. It is more common to refer to the German version, "Theorie der Einfachen Ungleichungen," which appeared in the *J. Reine Angew Math.*, 124, 1901, 1-27. For an exposition in English, see [3].

[2] However, I will note that the many proofs of the lemma via the separating hyperplane theorem 'cheat'. These proofs do not bother to show that the cone generated by the columns of A is a closed set.