Parametric Cardinality Probing in Set Partitioning

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Summary. In this work, we investigate parametric probing methods based on solution cardinality for set partitioning problems. The methods used are inspired by the early work of Gass and Saaty on the parametric solution to linear programs, as well as the later work of Joseph, Gass, and Bryson that examined the duality gap between the integer and relaxation solutions to general integer programming problems. Computational results are presented for a collection of set partitioning problems found in the literature.

Key words: Set partitioning; parametric cardinality probing.

1 Introduction

The set partitioning problem (SPP) assumes a set of m elements that are to be partitioned into mutually exclusive and collectively exhaustive subsets. In the enumeration of n possible subsets, one may define a matrix $[a_{ij}]$ where $a_{ij} = 1$ if the i-th element of the set is contained in the j-th subset, and $a_{ij} = 0$ otherwise. A set of binary decision variables associated with the n subsets is used in the model where $x_j = 1$ if subset j is used in the solution, and $x_j = 0$ otherwise. If the cost of creating each subset, j, is given as c_j, then the minimal cost set partitioning problem may be specified as the following binary linear program:

Minimize $\Sigma\, c_j\, x_j$,

Subject to: $\Sigma\, a_{ij}\, x_j = 1, i = 1, 2, \ldots, m$, and $a_{ij} = [0,1]$.

$x_j = [0,1], j = 1, 2, \ldots, n.$

The set partitioning model has been used in a wide variety of successful applications. The vehicle routing problem, for example, was first formulated as a set partitioning problem in 1964 [1]. Since that time, many researchers, see [21] for example, have used the model. Similarly, the airline crew scheduling problem was formally posed as a set partitioning problem in 1969 [1]. Subsequent solution approaches to the problem have included linear programming-based methods [22],

heuristic procedures [3], and column generation approaches [25]. Garfinkel and Nemhauser [6] considered the use of the set partitioning problem in the solution of political districting problems. This formulation has recently been solved effectively using branch and cut methods [24].

The cardinality of the solution of the set partitioning problem is equal to the number of subsets used in the solution. In the mathematical formulation of the problem given above, the cardinality of the solution may be determined by the summation of the x_j binary decision variables. This is simply:

$$\Sigma x_j = y,$$

where y is the cardinality of the solution. In cases where non-integer values of the decision variables are allowed, for example in the linear programming relaxation of the SPP, the value y will be called the global cover of the solution.

Set partitioning problems of a specified cardinality occur frequently in practice. In vehicle routing, for example, it may be desired to partition a set of m delivery customers into exactly k delivery routes. Similarly, in various crew scheduling applications where the number of crews is fixed, the m tasks to be scheduled are to be partitioned among the k crews. This is the case in the crew scheduling problems faced by many European airlines where the number of crews operating each month is fixed.

The closeness, numerically and structurally, of the integer and the associated linear relaxation solution to integer programs is frequently observed in the literature [12]. Consequently, many solution algorithms proposed for integer programming problems use the linear programming relaxation solution as their starting point. Typically, numerical closeness has been measured in terms of the gap between objective function values. Joseph, Gass and Bryson [19], examined the relationships between both the objective function values and the structure of the solution vectors for general integer programming problems and their respective relaxations. In the case of the SPP, it has been noted by various researchers, e.g. [11] and [23], that when the number of rows is small, linear programming, or linear programming with branch-and-bound, can provide integer solutions quickly. Further, it has been shown that if an application can be modeled as a set partitioning problem with a totally unimodular structure, then the linear programming solution is guaranteed to be integer [14], [15], and [16]. Finally, a decomposition approach, proposed by Joseph [18], was successful in quickly identifying optimal solutions to difficult set partitioning problems. The success of this approach was due in part to the inherent closeness of the relaxed and integer solution vectors so that enumerating a very small number of fractional variables gave rise to sub-problems that were solved in a fraction of the time it would have taken to solve the original set partitioning problem.

In this paper, we investigate the formal and explicit consideration of cardinality in the solution of minimal cost set partitioning problems. Specifically, we probe solution cardinality using linear programming techniques to search the global cover contours of the convex polytope as an approximation of the cardinality contours of the integer problem. The results of the cardinality probing are used to obtain