A Counting Problem in Linear Programming

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Summary. Using a popular setup, in solving a linear programming problem one looks for a tableau of the problem that has no negative elements in the last column and no positive elements in the last row. We study a matrix whose (i, j)–th entry counts the tableaux for such a problem (here taken to be totally nondegenerate) having i negative elements in the last column and j positive elements in the last row. It is shown that this matrix possesses a certain symmetry, which is described.

Key words: Linear programming; oriented matroid; tableau enumeration; polytope; h–vector; f–vector.

1 Introduction

It is assumed that the reader has some familiarity with linear programming. If not, there are a great many books that were written to serve as textbooks on linear programming; they are all, to some extent, descendants of the first such book written by Saul Gass [2].

Suppose we have a linear programming problem (assumed to be "totally nondegenerate"), having s nonnegative variables and r additional inequality constraints. In order to solve the problem using the simplex method, we may construct a suitable tableau and, by pivoting, attempt to move to a tableau in which the last column has no negative entries and the last row has no positive entries (ignoring the entry they share in common). A crude measure of progress toward this solution tableau is indicated by the pair of numbers (a, b), where a is the number of negative entries in the last column and b is the number of positive entries in the last row of the current tableau. Of course, these numbers may go up and down during the trek; it's not at all clear what this information tells us about getting to the solution. Even so, when these numbers aren't too big, intuition seems to dictate that we are "getting warmer," and that a tableau with $a = b = 0$ may not be many steps away. Thus we are led to the question of what can be said about the

$(r+1) \times (s+1)$ matrix N whose (a,b)–th entry (where $0 \le a \le r$, $0 \le b \le s$) is the number of tableaux for the problem having a negative entries in the last column and b positive entries in the last row. The main theorem below gives a simple property of this matrix.

This question is related to certain enumerative results concerning convex polytopes, arrangements of hyperplanes, and oriented matroids. Since our linear programming problem is totally nondegenerate, and under the assumption that the feasible region for the problem is nonempty and bounded, this feasible region is a simple convex polytope P of dimension s. ("Simple" means that, at each vertex, exactly s of the $r+s$ inequality constraints are satisfied with equality — each vertex lies on exactly s facets. Equivalently, exactly s edges emanate from each vertex.) Letting f_k denote the number of k–dimensional faces of P, so that f_0 is the number of vertices, f_1 is the number of edges, f_{s-1} is the number of facets, and $f_s = 1$, the vector $(f_0, \ldots, f_s)$ is the *f–vector of P*.

Each vertex of P is on exactly s edges, since P is simple. The objective function is not constant on any edge of this polytope, again by total nondegeneracy. We say that an edge that emanates from a vertex is *leaving* if the value of the objective function at that vertex is less than its value at the other vertex of the edge (so that the objective function increases when one moves along the edge away from the given vertex). Let h_j $(0 \le j \le s)$ denote the number of vertices of P for which there are exactly j leaving edges. The vector $(h_0, \ldots, h_s)$ is called the *h–vector.*

The f–vector can be determined from the h–vector, by the following system of linear equations:

$$f_k = \sum_{j=0}^{s} \binom{j}{k} h_j \qquad \text{for } 0 \le k \le s.$$

These equations result from the fact that each k–dimensional face of P possesses a unique vertex at which the objective function is minimized, and, if a given vertex has exactly j "leaving" edges, so that it contributes to h_j, then exactly $\binom{j}{k}$ faces of dimension k achieve their minima at the vertex.

Since the objective function factors into the determination of the h–vector in the above, one might guess that there would be lots of different h–vectors, depending upon the objective function chosen. In fact, there is only one, as can be seen by noting that the above system of equations is triangular with 1's on the diagonal, and therefore invertible. Upon inversion, it is clear that the h–vector can be determined from the f–vector. Since the f–vector does not depend upon the choice of objective function, any objective function (for which the nondegeneracy assumption is satisfied) yields the same h–vector. In particular, if the objective function is replaced by its negative, one sees that the h–vector is symmetric: $h_j = h_{s-j}$ $(0 \le j \le s)$. When these equations are written in terms of the f–vector, they are known as the *Dehn–Somerville equations.*