Toward Exposing the Applicability of Gass & Saaty's Parametric Programming Procedure

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Summary. In this paper we discuss some applications of Gass & Saaty's parametric programming procedure (GSP^3). This underutilized procedure is relevant to solution strategies for many problem types but is often not considered as a viable alternate approach. By demonstrating the utility of this procedure, we hope to attract other researchers to explore the use of GSP^3 as a solution approach for important problems in operations research, computer science, information systems, and other areas.

Key words: Parametric programming; integer programming; Lagrangian relaxation; multi-objective programming; multi-criteria decision making; simplex method; clustering; data mining; database systems.

1 Introduction

In this paper we present an overview of some applications of the parametric programming procedure of Gass and Saaty [16]. This procedure is relevant to solution strategies for many problem types but is often underutilized. In several cases it appears that it was not considered as an alternate approach, and the lack of its use was not justified. Our initial interest in applications of this procedure involved the Lagrangian relaxation of a network problem with a side constraint. This problem is a special case of the parametric linear programming problem (P^3); yet up to the mid-1980's it was not usually recognized as such and so was usually addressed using other techniques such as sub-gradient optimization. Bryson [6] demonstrated that this problem could be effectively addressed by the parametric programming procedure of Gass and Saaty (GSP^3). Since then we have observed other problems that could be addressed by this procedure. For example, Stanfel [32] addressed the sequential partitioning problem, which is a special case of the clustering problem, but did not recognize that this problem could also be addressed by GSP^3 [19]. More recently in Osei-Bryson and Joseph [30], we demonstrated that some technical information systems problems could be effectively addressed by GSP^3. In general we have seen special cases of P^3 in management science (MS), operations research (OR), computer science (CS), and information systems (IS) that were addressed by sub-optimal techniques. While we are not certain as to why this

situation occurs, some possible reasons include the researcher's unawareness of GSP^3 and/or lack of recognition of their research problem as being just a special case of P^3. Although in an earlier work [7], we discussed some applications of GSP^3, this situation suggests to us that it might be appropriate to re-tell at least part of the story on the wide applicability of GSP^3.

2 Gass & Saaty's Parametric Programming Procedure (GSP³)

The parametric programming procedure of Gass and Saaty (GSP^3) solves the following linear programming problem for every value of the scalar multiplier w:

$$P_1\text{: Min } \{Z_0(X) + wZ_1(X) \mid AX = b; X \geq 0\}$$

where $Z_0(X)$ (e.g., $CX = \sum_j c_j x_j$) and $Z_1(X)$ (e.g., $DX = \sum_j d_j x_j$) are linear functions of X, and w is a scalar multiplier. Using the simplex method, the parametric programming procedure first solves problem P_1 for $w = w_0$, where w_0 is some arbitrary small number. The result is that either there is some efficient extreme-point solution X_0, or that there is no finite minimum for $w = w_0$.

Let $J_{(k)}$ be the index set of basic variables associated with X_k, and let the reduced costs have the form $(\underline{c}_j + w\underline{d}_j)$, where $\underline{c}_j$ and $\underline{d}_j$ are the reduced costs in terms of $Z_0(X)$ and $Z_1(X)$, respectively.

For optimality we require that $(\underline{c}_j + w\underline{d}_j) \leq 0$. Therefore, X_k is the optimal solution of problem P_1 for the interval $[w_{k(L)}, w_{k(U)}]$ where $w_{k(L)}$, $w_{k(U)}$ are determined as follows:

$$\begin{aligned} w_{k(L)} &= -(\underline{c}_{k(L)}/\underline{d}_{k(L)}) = \text{Max } \{-\underline{c}_j/\underline{d}_j\text{: } \underline{d}_j < 0; j \notin J_{(k)}\}\text{; or} \\ &= -\infty \quad \text{if } \underline{d}_j \geq 0 \ \forall\ j \notin J_{(k)}. \end{aligned}$$

$$\begin{aligned} w_{k(U)} &= -(\underline{c}_{k(U)}/\underline{d}_{k(U)}) = \text{Min } \{-\underline{c}_j/\underline{d}_j\text{: } \underline{d}_j \geq 0; j \notin J_{(k)}\text{; or} \\ &= +\infty \quad \text{if } \underline{d}_j < 0 \ \forall\ j \notin J_{(k)}. \end{aligned}$$

The procedure is terminated if either of the following conditions occurs: (a) $w_{k(U)} = +\infty$, (b) $w_{k(U)}$ is finite but all the corresponding $a_{i,k(U)} \leq 0$. Otherwise, the simplex method introduces $x_{k(U)}$ into the basis and eliminates the basic variable in the usual manner. Gass and Saaty [16] established that the resulting basis yields a minimum for at least one value of w, and that if $[w_{k+1(L)}, w_{k+1(U)}]$ is the interval for which the resulting basis yields a minimum then $w_{k(U)} = w_{k+1(L)}$.

Thus, the parametric programming procedure generates the set of efficient, extreme points that solve the single parameter LP problem for all w such that $-\infty \leq w \leq +\infty$ and will identify the parametric interval $[w_{k(L)}, w_{k(U)}]$ that is associated with each such extreme point X_k.

The first step of GSP^3 involves solving the LP problem P_1 with $w = 0$. It is known that LP problems can be solved in polynomial time [21, 5], and so this first step can be done in polynomial time. Given this initial solution, the next solution