Twinless Strongly Connected Components

S. Raghavan

Robert H. Smith School of Business & Institute for Systems Research
Van Munching Hall
University of Maryland
College Park, MD 20742
raghavan@umd.edu

Summary. Tarjan [9] describes how depth first search can be used to identify Strongly Connected Components (SCC) of a directed graph in linear time. It is standard to study Tarjan's SCC algorithm in most senior undergraduate or introductory graduate computer science algorithms courses. In this paper we introduce the concept of a *twinless strongly connected component* (TSCC) of a directed graph. Loosely stated, a TSCC of a directed graph is (i) strongly connected, and (ii) remains strongly connected even if we require the deletion of arcs from the component, so that it does not contain a pair of *twin arcs* (twin arcs are a pair of bidirected arcs (i, j) and (j, i) where the tail of one arc is the head of the other and vice versa). This structure has diverse applications, from the design of telecommunication networks [7] to structural stability of buildings [8]. In this paper, we illustrate the relationship between 2-edge connected components of an undirected graph—obtained from the strongly connected components of a directed graph—and twinless strongly connected components. We use this relationship to develop a linear time algorithm to identify all the twinless strongly connected components of a directed graph. We then consider the augmentation problem, and based on the structural properties developed earlier, derive a linear time algorithm for the augmentation problem.

Key words: Digraph augmentation; strong connectivity; linear time algorithm.

1 Introduction

Let $D = (N, A)$ be a directed graph (digraph) with node set N and arc set A. A pair of nodes x and y are *twinless reachable* if there exists a directed path from node x to node y, and a directed path from node y to node x, such that for every arc (i, j) contained in the path from node x to node y, the path from node y to node x does not contain arc (j, i). The *Twinless Strongly Connected Components* (TSCCs) of a digraph are the equivalence classes of nodes under the "twinless reachable" condition (we will show later that the twinless reachable condition defines an equivalence relationship). We

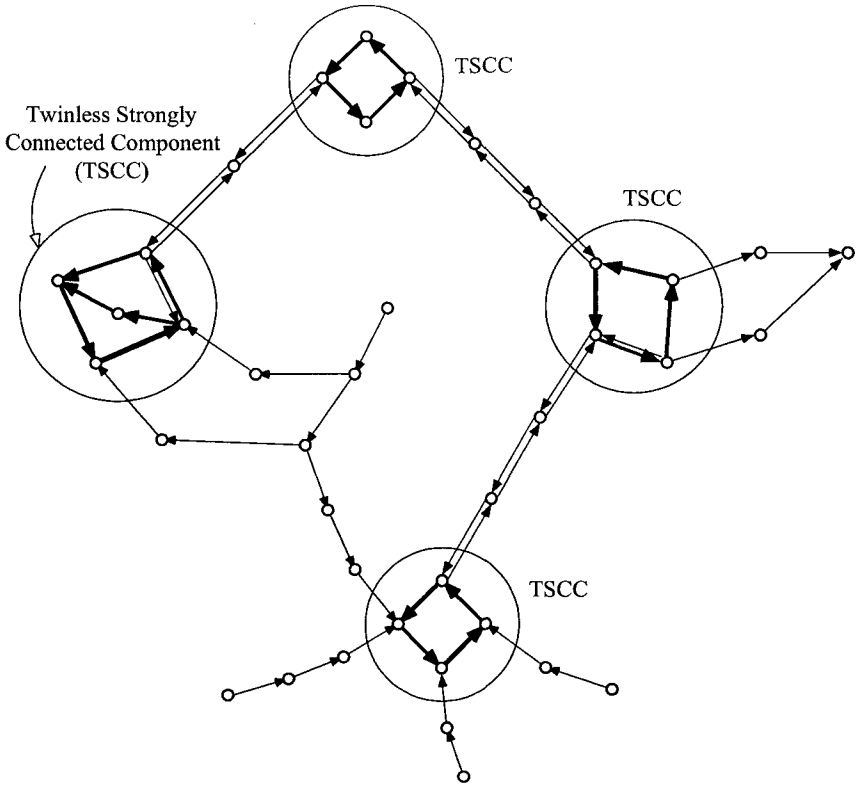

Fig. 1. Twinless Strongly connected components of a digraph. Bold arcs show twinless arcs that form a strongly connected component.

say a digraph is *Twinless Strongly Connected* if every pair of nodes is twinless reachable.

We now provide a slightly different, but equivalent, definition of twinless strongly connectedness. We say that a pair of bidirected arcs (i, j) and (j, i) are *twins*. Recall that a digraph is strongly connected if it contains a directed path between every pair of its nodes. Our alternate definition then is as follows. A digraph $D = (N, A)$ is *Twinless Strongly Connected* if for some subset A' of A, the digraph (N, A') is strongly connected and A' does not contain an arc together with its twin. A *Twinless Strongly connected component* (TSCC) of a digraph is the node set of a maximal twinless strongly connected subdigraph of D. Figure 1 gives an example of four TSCCs, that contain 3 or more nodes, in a digraph.

It should be apparent that every pair of nodes in a TSCC, as defined by the second definition, are twinless reachable. What may not readily apparent is the