EOQ Rides Again!

Beryl E. Castello and Alan J. Goldman

Department of Applied Mathematics and Statistics
The Johns Hopkins University
Baltimore, MD 21218
castello@ams.jhu.edu, goldman@ams.jhu.edu

Summary. Despite the criticisms of Woolsey [25] and others, the classical "EOQ" model of Inventory Theory remains useful, both pedagogically and as a stepping-stone towards more realistic variants. Prominent among these are: (a) the one in which stock-outs are permitted but penalized, and (b) the one in which deliveries are gradual (i.e., continuous) rather than instantaneous. Here we also deal with a variant containing both features, (a) and (b). (In addition, a less common variant is considered.) In the spirit of Harris's original (1913) presentation, the use of calculus is avoided entirely. Instead, it is shown explicitly how all the variants can be solved by "reduction" to the basic model, which is in turn treated by using the simply-proved "base case" of the arithmetic-geometric mean inequality.

Key words: Economic order/production quantity; inventory management; shortage; algebra; EOQ; EPQ.

1 Introduction

The setting for an inventory cost analysis can be described as follows: the decision maker, on whose behalf the optimization is conducted, can be thought of as the manager of a firm which sells a product. The firm replenishes its stock by reordering more, from either an external wholesaler or a capability for manufacturing the goods internally. The decision maker must determine (1) how often to reorder, and (2) the size of the reorder quantity.

The usual objective of the analysis is to minimize the total cost per unit time associated with inventory. This is comprised of: (1) holding costs (the costs of maintaining or "carrying" inventory), (2) reordering or production costs (the "fixed" costs from individual replenishments of inventory), (3) procurement costs (the "variable" costs of purchasing each unit of inventory), and (4) shortage costs (the cost of subjecting customers to delays from a backlog of unfilled demand).

Inventory models have been studied extensively since Harris first presented the famous economic order quantity (EOQ) formula in 1913 ([17]; see [11] for

historical background). Until recently, almost all the literature concerning the EOQ and related inventory models[1] used differential calculus as the underlying solution technique, though frequently omitting the supplementary reasoning needed to establish optimality. (For example, see [8], [16], [3], [1], [21], and [10].) One early exception occurred in 1970, when Thierauf and Grosse [24] developed the optimal order quantity for the EOQ model using three different techniques: graphical analysis, algebraic analysis (which is later used to solve the EPQ model), and calculus. The authors based their algebraic method on the presumed validity of equating opposing costs (i.e., holding costs versus ordering costs).

In 1996, Grubbström [14] derived the EOQ formula without the use of calculus via an elegant "completing the square" argument. Grubbström and Erdem [15] then obtained the solution to an EOQ model with shortages without the use of derivatives, and proposed that their approach be used to introduce students without a background in calculus to inventory models. In 2001, Cárdenas-Barrón [4] revisited the work of Grubbström and Erdem and extended it to the economic production quantity (EPQ) model with shortages. In 2004, Ronald et al. [22] suggested that the method presented in the previous two articles might still be too sophisticated for some readers; they provided rationale (derivation rather than mere verification) for some of the identities used in [15] and [4]. A 2005 article by Chang et al. [5] attempted to simplify the algebra of the previous three articles, such that "the usual skill of completing the square" could more readily handle the objective functions for the EOQ and EPQ models with shortages.

In this work, we derive optimal solutions for four inventory models – EOQ, EPQ, EOQ with shortages, and EPQ with shortages – again without the use of derivatives. In each case, we begin with a step-by-step development of the objective function. To keep the methods entirely elementary, our approach uses the easily-proved "base case" of the arithmetic-geometric mean inequality to identify the optimal solution for the classical EOQ model. We then show that the three variants can be reduced to this basic EOQ model. The final section includes a similar treatment for a less common variant (see [12]) involving "deferred payment" for the delivered goods; that treatment can also be adapted to Chung and Huang's [7] extension of Goyal's variant from EOQ to EPQ.

[1] Literature-tracing here has been hindered considerably because different authors use different names and/or acronyms for the same scenario, and vice versa. For example, the economic production quantity (EPQ) model is sometimes called the "(economic) production lot size model." Also, the term "economic lot size" has been used for both the EOQ and EPQ models.