Chapter 9
Lipschitz and $BMO$ norms

In this chapter we provide some norm comparison theorems related to the $BMO$ norms and the Lipschitz norms. We prove that the integrability exponents described in the Lipschitz norm comparison theorem (Theorem 9.2.1) are the best possible. We also develop some norm comparison theorems for the operators.

9.1 Introduction

The bounded mean oscillation ($BMO$) space was introduced by John and Nirenberg in 1961. “Bounded mean oscillation” soon became one of the main concepts in many fields, such as harmonic analysis, complex analysis, and partial differential equations. A function $f \in L_{loc}^1(\Omega, \mu)$ is said to be in $BMO(\Omega, \mu)$ if there is a constant $C$ such that

$$\frac{1}{\mu(B)} \int_B |f - f_B| d\mu \leq C \quad (9.1.1)$$

for all balls $B$ with $\sigma B \subset \Omega$, where $\sigma > 1$ is a constant. The least $C$ for which (9.1.1) holds is denoted by $\|f\|_* = \|f\|_{*,\Omega}$ and called the $BMO$ norm of $f$. Equivalently,

$$\|f\|_{*,\Omega} = \sup_{\sigma B \subset \Omega} \frac{1}{\mu(B)} \int_B |f - f_B| d\mu, \quad (9.1.2)$$

where $\sigma > 1$ is a constant.

One of the most useful results for $BMO$ is the John–Nirenberg lemma which was first proved by Muckenhoupt and Wheeden in [146]. Abstract versions of the John–Nirenberg lemma are also available, see [313], for example.
For relationship between $BMO$ and quasiconformal mappings, see [70, 71].

Also, see [123] for properties of $BMO$ spaces.

**Theorem 9.1.1. (John–Nirenberg lemma for doubling weights).** Let $\mu$ be defined by $d\mu = w(x)dx$ and $w(x)$ be a doubling weight. Then, a function $f$ is in $BMO(\Omega, \mu)$ if and only if

$$\mu(\{x \in B : |f(x) - f_B| > t\}) \leq c_1 e^{-c_2 t} \mu(B)$$  \hspace{1cm} (9.1.3)

for each ball $B \subset \Omega$ and $t > 0$. Here $c_1$ and $c_2$ are positive constants.

The proof of John–Nirenberg lemma is also available in [59] where the Calderón–Zygmund decomposition technique is adopted. We state the following corollary from [59] which provides a useful tool to study the averaging domains.

**Corollary 9.1.2.** A function $f$ is in $BMO(\Omega, \mu)$ if and only if there exist positive constants $k$ and $C$ such that

$$\frac{1}{\mu(B)} \int_B e^{k|f-f_B|}d\mu \leq C$$  \hspace{1cm} (9.1.4)

for any ball $B$ which is a compact subset of $\Omega$. If (9.1.4) holds, then $\|f\|_* \leq C/k$. Conversely, if $f$ is in $BMO(\Omega, \mu)$, then (9.1.4) holds with $C = 3$ and $k = (\log 2)/(8c_0N\|f\|_*)$, where $N$ and $c_0$ are constants appearing in the Calderón–Zygmund decomposition for doubling weights.

**9.2 BMO spaces and Lipschitz classes**

In this section, we will first present some interesting results for the $BMO$ spaces and the Lipschitz spaces of differential forms, and, then provide an example to show that the integrability exponents in Lipschitz conditions for conjugate $A$-harmonic tensors are the best possible.

**9.2.1 Some recent results**

Let $\omega \in L^1_{loc}(\Omega, \wedge^l)$, $l = 0, 1, \ldots, n$. We write $\omega \in \text{locLip}_k(\Omega, \wedge^l)$, $0 \leq k \leq 1$, if

$$\|\omega\|_{\text{locLip}_k, \Omega} = \sup_{\sigma Q \subset \Omega} |Q|^{-(n+k)/n} \|\omega - \omega_Q\|_{1, Q} < \infty$$  \hspace{1cm} (9.2.1)