Chapter 2

Inversion Formulae and Practical Results

2.1 The Uniqueness Property

We mentioned in the last Chapter that the Laplace transform is unique in
the sense that if \( \bar{f}(s) = \bar{g}(s) \) and \( f(t) \) and \( g(t) \) are continuous functions then
\( f(t) = g(t) \). This result was proved originally by Lerch [125] and the proof given
here follows that in Carslaw and Jaeger [31].

**Theorem 2.1 (Lerch’s theorem).**

If

\[
\bar{f}(s) = \int_{0}^{\infty} e^{-st} f(t) dt, \quad s > \gamma, \tag{2.1}
\]

is satisfied by a continuous function \( f(t) \), there is no other continuous function
which satisfies the equation (2.1).

**Proof.** We require the following lemma.

**Lemma 2.1** Let \( \psi(x) \) be a continuous function in \([0, 1]\) and let

\[
\int_{0}^{1} x^{n-1} \psi(x) dx = 0, \quad \text{for} \quad n = 1, 2, \ldots. \tag{2.2}
\]

Then

\[
\psi(x) \equiv 0, \quad \text{in} \quad 0 \leq x \leq 1. \tag{2.3}
\]

**Proof.** If \( \psi(x) \) is not identically zero in the closed interval \([0, 1]\), there must be
an interval \([a, b]\) where \( 0 < a < b < 1 \) in which \( \psi(x) \) is always positive (or always
negative). We shall suppose the first alternative. By considering the function
\((b-x)(x-a)\) we see that if

\[
c = \max[ab, (1-a)(1-b)],
\]
then
\[ 1 + \frac{1}{c} (b - x)(x - a) > 1, \quad \text{when} \ a < x < b \]
and
\[ 0 < 1 + \frac{1}{c} (b - x)(x - a) < 1, \quad \text{when} \ 0 < x < a \ \text{and} \ b < x < 1. \]

Thus the function
\[ p(x) = \{1 + (1/c)(b - x)(x - a)\}^r, \]
can be made as large as we please in \( a < x < b \) and as small as we like in \( 0 < x < a, b < x < 1 \) by appropriate choice of \( r \). But \( p(x) \) is a polynomial in \( x \), and by our hypothesis
\[ \int_0^1 x^{n-1} \psi(x) \, dx = 0, \quad \text{for} \ n = 1, 2, \cdots \]
we should have
\[ \int_0^1 p(x) \psi(x) \, dx = 0, \]
for every positive integer \( r \). But the above inequalities imply that, by choosing \( r \) large enough,
\[ \int_0^1 p(x) \psi(x) \, dx > 0. \]
The first alternative thus leads to a contradiction. A similar argument applies if we assume \( \psi(x) < 0 \) in \((a,b)\). It therefore follows that \( \psi(x) \equiv 0 \) in \([0,1] \). ■

Now suppose that \( g(t) \) is another continuous function satisfying (2.1) and define \( h(t) = f(t) - g(t) \) which, as the difference of two continuous functions, is also continuous. Then
\[ \int_0^\infty e^{-st} h(t) \, dt = 0, \quad s \geq \gamma. \quad (2.4) \]
Let \( s = \gamma + n \), where \( n \) is any positive integer. Then
\[ \int_0^\infty e^{-(\gamma+n)t} h(t) \, dt = \int_0^\infty e^{-nt} \left( e^{-\gamma t} h(t) \right) \, dt, \]
\[ = \left[ e^{-nt} \int_0^t e^{-\gamma u} h(u) \, du \right]_0^\infty + n \int_0^\infty e^{-nt} \left[ \int_0^t e^{-\gamma u} h(u) \, du \right] \, dt, \]
\[ = n \int_0^\infty e^{-nt} \left[ \int_0^t e^{-\gamma u} h(u) \, du \right] \, dt, \]
and thus it follows from (2.4) that
\[ \int_0^\infty e^{-nt} \phi(t) \, dt = 0, \]