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Reconstruction With Orthogonal Functions

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1. INTRODUCTION

In 1917, Johann Radon posed the question of whether the integral over a function with two variables along an arbitrary line can uniquely define that function such that this functional transformation can be inverted. He also solved this problem as a purely mathematical one, although he mentioned some relationships to the physical potential theory in the plane. Forty-six years later, A. M. Cormack published a paper with a title very similar to that by Radon yet still not very informative to the general reader, namely 'Representation of a function by its line integrals'—but now comes the point: 'with some radiological applications'. Another point is that the paper appeared in a journal devoted to applied physics. Says Cormack, 'A method

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is given of finding a real function in a finite region of a plane given by its line integrals along all lines intersecting the region. The solution found is applicable to three problems of interest for precise radiology and radiotherapy'. Today we know that the method is useful and applicable to the solution of many more problems, including that which won a Nobel prize in medicine, awarded to A. M. Cormack and G. N. Hounsfield in 1979. Radon's pioneering paper (1917) initiated an entire mathematical field of integral geometry. Yet it remained unknown to the physicists (also to Cormack, whose paper shared the very same fate for a long time). However, the problem of projection and reconstruction, the problem of tomography as we call it today, is so general and ubiquitous that scientists from all kinds of fields stumbled on it and looked for a solution—without, however, looking back or looking to other fields. Today there is a vast literature which cannot comprehensively be appreciated in this short contribution. It was Cormack (1963, 1964) who first made use of orthogonal functions for the solution of Radon's problem. Not only is their application elegant, but it also provides a good understanding about the inner relationships of a structure to its projections. The goal of this contribution is to demonstrate these relationships.

2. ORTHOGONAL POLYNOMIALS

As we do not presuppose any knowledge about orthogonal polynomials, we will compile a certain minimum which, however, is derived from knowledge certainly available to anybody engaged in advanced electron microscopy, diffraction theory and Fourier techniques. We are concerned with the properties of orthogonal polynomials, but abstain from general derivations. Rather, we demonstrate them for the special functions that are needed in tomography and point out that their properties are generally valid for all orthogonal polynomials. Without calculation, we all know that the integral over a periodic function vanishes when a number of plus and minus cycles proper for a complete cancelation fall within the range of integration

$$\int_0^{\pi} \cos n\phi \, d\phi = 0. \tag{1}$$

Only if we change n, the number of nodes (zeros) within the interval, to nil does the integration result in a finite value equal to π. However, actually no node means also that we violate the original agreement of periodic functions. One can retain the periodicity and yet avoid the cancellations by squaring the integrand; and, again without calculation and just by reasoning, one can arrive at

$$\int_0^{\pi} \cos^2 n\phi \, d\phi = \frac{\pi}{2}; \quad n \neq 0.$$

Multiplication of two such harmonic functions leads to