Chapter 2
The Uniform Boundedness Principle and the Closed Graph Theorem

2.1 The Baire Category Theorem

The following classical result plays an essential role in the proofs of Chapter 2.

• Theorem 2.1 (Baire). Let $X$ be a complete metric space and let $(X_n)_{n \geq 1}$ be a sequence of closed subsets in $X$. Assume that

$$\text{Int } X_n = \emptyset \quad \text{for every } n \geq 1.$$  

Then

$$\text{Int } \left( \bigcup_{n=1}^{\infty} X_n \right) = \emptyset.$$  

Remark 1. The Baire category theorem is often used in the following form. Let $X$ be a nonempty complete metric space. Let $(X_n)_{n \geq 1}$ be a sequence of closed subsets such that

$$\bigcup_{n=1}^{\infty} X_n = X.$$  

Then there exists some $n_0$ such that $\text{Int } X_{n_0} \neq \emptyset$.

Proof. Set $O_n = X_n^c$, so that $O_n$ is open and dense in $X$ for every $n \geq 1$. Our aim is to prove that $G = \bigcap_{n=1}^{\infty} O_n$ is dense in $X$. Let $\omega$ be a nonempty open set in $X$; we shall prove that $\omega \cap G \neq \emptyset$.

As usual, set

$$B(x, r) = \{ y \in X ; \; d(y, x) < r \}.$$  

Pick any $x_0 \in \omega$ and $r_0 > 0$ such that

$$B(x_0, r_0) \subset \omega.$$  

Then, choose $x_1 \in B(x_0, r_0) \cap O_1$ and $r_1 > 0$ such that

$$B(x_1, r_1) \subset \omega.$$  

which is always possible since \( O_1 \) is open and dense. By induction one constructs two sequences \((x_n)\) and \((r_n)\) such that

\[
\begin{cases}
B(x_{n+1}, r_{n+1}) \subset B(x_n, r_n) \cap O_{n+1}, \\
0 < r_{n+1} < \frac{r_n}{2}.
\end{cases}
\]

It follows that \((x_n)\) is a Cauchy sequence; let \( x_n \to \ell \).

Since \( x_{n+p} \in B(x_n, r_n) \) for every \( n \geq 0 \) and for every \( p \geq 0 \), we obtain at the limit (as \( p \to \infty \)),

\[ \ell \in \overline{B(x_n, r_n)}, \quad \forall n \geq 0. \]

In particular, \( \ell \in \omega \cap G \).

\[ 2.2 \text{ The Uniform Boundedness Principle} \]

**Notation.** Let \( E \) and \( F \) be two n.v.s. We denote by \( \mathcal{L}(E, F) \) the space of continuous (= bounded) linear operators from \( E \) into \( F \) equipped with the norm

\[ \| T \|_{\mathcal{L}(E,F)} = \sup_{\| x \| \leq 1} \| Tx \|. \]

As usual, one writes \( \mathcal{L}(E) \) instead of \( \mathcal{L}(E, E) \).

**Theorem 2.2 (Banach–Steinhaus, uniform boundedness principle).** Let \( E \) and \( F \) be two Banach spaces and let \( (T_i)_{i \in I} \) be a family (not necessarily countable) of continuous linear operators from \( E \) into \( F \). Assume that

\[
\begin{align*}
(1) & \quad \sup_{i \in I} \| T_i x \| < \infty, \quad \forall x \in E, \\
(2) & \quad \sup_{i \in I} \| T_i \|_{\mathcal{L}(E,F)} < \infty.
\end{align*}
\]

Then

In other words, there exists a constant \( c \) such that

\[ \| T_i x \| \leq c \| x \|, \quad \forall x \in E, \quad \forall i \in I. \]

**Remark 2.** The conclusion of Theorem 2.2 is quite remarkable and surprising. From pointwise estimates one derives a global (uniform) estimate.

**Proof.** For every \( n \geq 1 \), let

\[ X_n = \{ x \in E; \quad \forall i \in I, \quad \| T_i x \| \leq n \}, \]