2 Geometrical quantities and series in Leibniz

Leibniz began to study higher mathematics in 1672 following Huygens’ prompting. In a short time he became interested in infinite series. It is well-known that the investigation of number sequences and their difference and sum sequences was of great importance in his discovery of the calculus.\(^{39}\) He himself asserted

\[
\text{I arrived at the method of inassignables through the method of infinite increments in the series of numbers, as the nature of the things requires. (Leibniz [GMS, 4:413])}
\]

Between 1675 and 1676, he wrote a treatise, entitled \textit{De quadratura arithmetica circuli ellipses et hyperbolae cujus corollarium est trigonometria sine tabulis} [KQA], which aimed to provide the quadrature of certain curves by means of series. Leibniz wrote at least six versions of this treatise, which nevertheless remained unpublished through his life. Only recently has Eberhard Knobloch published its last and most extensive version, which consists of 51 propositions and many scholia.\(^{40}\) In the years that followed, Leibniz published many of the results of the \textit{De quadratura arithmetica} (but not the proofs and the solution methods) in \textit{De vera proportione circuli ad quadratum circumscrip tum in numeris rationalibus expressa} [1682] and in \textit{Quadratura arithmetica communis sectionum conicarum} [1691].

Then Leibniz wrote other important papers concerning series, in particular \textit{Supplementum geometriae practicae sese ad problema transcendentia extendens, ope novae methodi generalissimae per series infinitas} [1693] and \textit{Epistola ad V. Cl. Christianum Wolfium Grandi} [1710].

This chapter is divided into two sections. The first is devoted to the investigation of Leibniz’s notion of convergence and the way in which he manipulated series. In the second section, I shall examine how power series were employed in the geometrical context of the early calculus. Other aspects of Leibniz’s conception (Leibniz’s derivation of the Bernoulli series, Leibniz’s analogy, the rise of the question of divergent series) are discussed in Chapters 3 and 9.

2.1 The capacity of series to express quantities and their manipulation

In \textit{De quadratura arithmetica}, Leibniz formulated the widely-known theorem on the sum of a geometric series as follows:

\[
\text{The greatest term of an infinite geometric series is the mean proportional between the greatest sum and the greatest difference. (see Leibniz [KQA, 71])}
\]

\(^{39}\)See Bos [1974, 13] and Guicciardini [1999, 137–138].

\(^{40}\)For an analysis of the manuscript, see Knobloch [1989] and [1991].
Leibniz’s proof is geometric and runs as follows. Given a decreasing geometric series $a_n$ and a straight line $s$, one draws a segment $A_1B_1$ so that it equals the first term $a_1$ of the series and is perpendicular to $s$ (see Fig. 5). One takes $A_2$ on the line $s$ such that $A_1A_2 = A_1B_1$ and draws the segment $A_2B_2$ so that it equals the second term $a_2$ of the series and is perpendicular to $s$. Then one takes $A_3$ on $s$ such that $A_2A_3 = A_2B_2$ and so on. The point $B_n$ fall on the line $B_1B_2$, which intersects $s$ in $C$. Leibniz states that the segment $A_1C$ is the sum of the series. Indeed the triangles $D_nB_nB_{n+1}$ are similar to $A_nB_nC$, and therefore the segments $A_nC$ are proportional to

$$D_nB_{n+1} = A_nA_{n+1} = A_nB_n.$$ 

It is true that no $A_n$ coincides with $C$; however, the segments $A_nB_n$ become smaller than any quantity and the sequence $A_n$ approaches $C$ closer and closer, with an error less than any assignable quantity. Since

$$A_1C : A_1B_1 = A_1B_1 : D_1B_1,$$

one has

$$S : a_1 = a_1 : (a_1 - a_2),$$

where $S$ is the sum of the series (see Leibniz [KQA, 71–73]).

Leibniz, like the other mathematicians considered in chapter 1, thought that the sum of a series was a determinate quantity (the segment $A_1C$) to which the partial sums of the series approached increasingly. Leibniz’s