Chapter 4
Markov Chains at Equilibrium

4.1 Introduction
In this chapter we will study the long-term behavior of Markov chains. In other words, we would like to know the distribution vector $s(n)$ when $n \to \infty$. The state of the system at equilibrium or steady state can then be used to obtain performance parameters such as throughput, delay, loss probability, etc.

4.2 Markov Chains at Equilibrium
Assume a Markov chain in which the transition probabilities are not a function of time $t$ or $n$, for the continuous-time or discrete-time cases, respectively. This defines a homogeneous Markov chain. At steady state as $n \to \infty$ the distribution vector $s$ settles down to a unique value and satisfies the equation

$$Ps = s \quad (4.1)$$

This is because the distribution vector value does not vary from one time instant to another at steady state. We immediately recognize that $s$ in that case is an eigenvector for $P$ with corresponding eigenvalue $\lambda = 1$. We say that the Markov chain has reached its steady state when the above equation is satisfied.

4.3 Significance of $s$ at “Steady State”
Equation (4.1) indicates that if $s$ is the present value of the distribution vector, then after one time step the distribution vector will be $s$ still. The system is now in equilibrium or steady state. The reader should realize that we are talking about probabilities here.

At steady state the system will not settle down to one particular state, as one might suspect. Steady state means that the probability of being in any state will not change with time. The probabilities, or components, of the vector $s$ are the ones that
are in steady state. The components of the transition matrix $P^n$ will also reach their steady state. The system is then said to be in steady state.

Assume a five-state system whose equilibrium or steady-state distribution vector is

$$s = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \end{bmatrix}^t$$

(4.2)

$$s = \begin{bmatrix} 0.2 & 0.1 & 0.4 & 0.1 & 0.2 \end{bmatrix}^t$$

(4.3)

Which state would you think the system will be in at equilibrium? The answer is, the system is in state $s_1$ with probability 20%, or the system is in state $s_2$ with probability 10%, etc. However, we can say that at steady state the system is most probably in state $s_3$ since it has the highest probability value.

### 4.4 Finding Steady-State Distribution Vector $s$

The main goal of this chapter is to find $s$ for a given $P$. Knowledge of this vector helps us find many performance measures for our system such as packet loss probability, throughput, delay, etc. The technique we choose for finding $s$ depends on the size and structure of $P$.

Since the steady-state distribution is independent of the initial distribution vector $s(0)$, we conclude therefore that $P^n$ approaches a special structure for large values of $n$. In this case we find that the columns of $P^n$, for large values of $n$, will all be identical and equal to the steady-state distribution vector $s$. We could see that in Examples 3.11 on page 82 and Example 3.20 on page 100.

**Example 4.1** Find the steady-state distribution vector for the given transition matrix by

(a) calculating higher powers for the matrix $P^n$
(b) calculating the eigenvectors for the matrix.

$$P = \begin{bmatrix} 0.2 & 0.4 & 0.5 \\ 0.8 & 0 & 0.5 \\ 0 & 0.6 & 0 \end{bmatrix}$$

The given matrix is column stochastic and hence could describe a Markov chain. Repeated multiplication shows that the entries for $P^n$ settle down to their stable values.