In this chapter we discuss some known constructions of unitals embedded in non-Desarguesian projective planes of square order, some of which are translation planes and some of which are not. As discussed in Section 4.1, ovoidal-Buekenhout-Metz unitals exist in any two-dimensional translation plane, and nonsingular-Buekenhout unitals exist in certain derivable two-dimensional translation planes. Here we discuss a wide variety of constructions, not just the Buekenhout techniques. We also show that it is possible for a Buekenhout unital to be embedded in two nonisomorphic planes.

5.1 Unitals in Hall Planes

The Hall plane of order $q^2$ can be obtained by deriving the Desarguesian plane $\text{PG}(2, q^2)$ of order $q^2$ as discussed in Section 3.2. We review the notation here. Let $D$ be a derivation set of $\ell_\infty$ in $\text{PG}(2, q^2)$. That is, $D$ is a Baer subline of $\ell_\infty \cong \text{PG}(1, q^2)$. The Hall plane, $\text{Hall}(q^2)$, has as points the points of $\text{PG}(2, q^2) \setminus D$ and $q + 1$ new points denoted by $D'$. The lines of $\text{Hall}(q^2)$ are of three types: lines of $\text{PG}(2, q^2)$ that meet $\ell_\infty$ in a point not in $D$; Baer subplanes of $\text{PG}(2, q^2)$ that contain $D$; the line at infinity, $\ell'_\infty$, which consists of the points of $\ell_\infty \setminus D$ and the points in $D'$ (the latter points correspond to the new slope points). Incidence in $\text{Hall}(q^2)$ is inherited from $\text{PG}(2, q^2)$. Of course, we may choose $\ell_\infty$ to be any line of $\text{PG}(2, q^2)$.

Thus, as shown in Section 3.4.3, we may assume the following Bruck-Bose representation for the Hall plane. Let $\mathcal{S}$ be a regular spread in the hyperplane $\Sigma_\infty$ of $\text{PG}(4, q)$, so that $\mathcal{P}(\mathcal{S})$ is the Desarguesian plane $\text{PG}(2, q^2)$. Let $\mathcal{R}$ be a regulus of $\mathcal{S}$ with opposite regulus $\mathcal{R}'$. Then $\mathcal{S}' = (\mathcal{S} \setminus \mathcal{R}) \cup \mathcal{R}'$ is a spread of $\Sigma_\infty$ that constructs the Hall plane; that is, $\mathcal{P}(\mathcal{S}') \cong \text{Hall}(q^2)$. Hence the Hall plane is a translation plane of dimension two over its kernel. This plane is isomorphic to the one coordinatized by the Hall quasifield. See [140] for more details on the Hall plane.
Recall that the only known unitals embedded in the Desarguesian plane are Buekenhout unitals, and the only nonsingular-Buekenhout unital embedded in the Desarguesian plane is the classical unital. If \( U \) is any unital embedded in \( \text{PG}(2, q^2) \), then when we derive this Desarguesian plane with respect to a derivation set \( D \) on \( \ell_\infty \), we obtain a corresponding set of points \( U' \) in \( \text{Hall}(q^2) \). To define \( U' \) more precisely, let \( \text{aff} U \) be the affine points of \( U \); that is, \( \text{aff} U = U \setminus \ell_\infty \) (that is, \( U \) minus the points of \( U \) that lie on \( \ell_\infty \)). Since the affine points of \( \text{PG}(2, q^2) \) are the same as the affine points of \( \text{Hall}(q^2) \), we have \( \text{aff} U' = \text{aff} U \) in \( \text{Hall}(q^2) \). We complete \( \text{aff} U' \) to a set \( U' \) in \( \text{Hall}(q^2) \) by adding the points on the line at infinity \( \ell'_\infty \) of \( \text{Hall}(q^2) \) which lie on a \( q \)-secant of \( \text{aff} U' \). We want to determine when the set \( U' \) is a unital in \( \text{Hall}(q^2) \).

We begin by focusing on the classical unital embedded in \( \text{PG}(2, q^2) \). Every chord of a classical unital is a Baer subline and so can serve as a derivation set. Hence there are five different ways to position the derivation set in relation to a classical unital \( U \) embedded in \( \text{PG}(2, q^2) \); these cases are illustrated in Figure 5.1.

![Fig. 5.1. Deriving the classical unital in \( \text{PG}(2, q^2) \)](image)

We first consider the case when \( U \) is secant to \( \ell_\infty \). Then \( U \) corresponds to a nonsingular quadric \( U \) in \( \text{PG}(4, q) \) that meets \( \Sigma_\infty \) in a hyperbolic quadric which is ruled by a regulus \( R \) of the regular spread \( S \). Of course, the opposite regulus \( R' \) also rules this hyperbolic quadric. Now \( D = U \cap \ell_\infty \) is a Baer subline, and thus is a derivation set. If we derive \( \text{PG}(2, q^2) \) with respect to \( D \), then we obtain the Hall plane which is constructed in the Bruck-Bose representation by the spread \( S' = (S \setminus R) \cup R' \). The points of \( U' \) in \( \text{Hall}(q^2) \) correspond to the same nonsingular quadric \( U \), which “meets” the spread \( S' \).