APPENDIX 6

EQUATIONAL COMPACTNESS

By Günter H. Wenzel

In the beginning there were the algebraically compact Abelian groups as introduced by I. Kaplansky in the 1954 edition of his monograph Infinite Abelian groups (I. Kaplansky [1954]): An Abelian group is algebraically compact if it has the form \( G = C \oplus D \) where \( C \) is divisible and \( D \) is a complete direct sum of groups \( D_p \), one for each prime \( p \); \( D_p \) is a module over the \( p \)-adic integers with no elements of infinite height and it is complete in its \( p \)-adic topology. The 1969 edition of the same book gives a quite different definition, reflecting a development that originated in Poland. S. Balcerzyk [1957] and J. Łoś [1957] discovered that Kaplansky’s rather involved structural definition is equivalent to either one of the following two: (1) Every finitely solvable system of algebraic equations over the given group \( G \) is solvable (in \( G \)). (2) \( G \) is a direct summand of a compact topological group.

J. Mycielski [3] gave the concept its proper universal algebraic setting and initiated a series of investigations dealing with the new topic. We shall restrict ourselves to what we believe are the central results from a universal algebraic standpoint. We cannot even attempt to cover all aspects or all results. The proofs will be concise, some will be omitted; nevertheless we hope to pass on the flavor of the subject.

§84. EQUATIONAL AND ATOMIC COMPACTNESS—
FIRST EXAMPLES

Equational compactness represents a successful attempt to carry over properties and methods of the theory of compact topological spaces to certain algebraic model theoretic questions. Some fascinating behavior of classical arithmetical structures on the borderline of algebra and naive set theory on the one hand, some surprising interplays between topology, equational compactness, algebra and model theory in important classes of algebraic systems on the other hand have created increasing interest in the field. The Zariski topology in affine or projective spaces of algebraic geometry can be considered as possibly the first realization of this pheno-
menon: Solution sets of polynomial equations are made the subbase of closed sets of a topology—in spite of the fact that this topology becomes very weak and atypical from a topologist's point of view. Of course, the analogy ends quickly. There we have quasi-compactness immediately (the Zariski topologies over algebraically closed fields are Noetherian with respect to open sets). With regard to the algebraic questions it is here that the problem begins: When do the solution sets of algebraic equations “behave quasi-compact”? What does such behavior mean for individual algebras? What does it mean for classes or—of primary interest—for equational classes? The compactness theorem and its various generalizations offer interesting problems of their own.

To illustrate the ideas we begin with a few examples. We start with a precise definition of equational compactness.

**Definition 1.** (1) An algebra \( A = \langle A; F \rangle \) is equationally compact if every infinite system \( \Sigma \) of algebraic equations over \( A \) in the variables \( x_\gamma \), \( \gamma < \beta \), is simultaneously solvable in \( A \) provided that every finite subset has a simultaneous solution. An algebraic equation over \( A \) is an equation \( p = q \) where \( p, q \) are algebraic functions of \( A \) (i.e., polynomials with constants from \( A \)). We consider the solution sets in \( A^\beta \) (or in some \( A^\gamma \) for \( \gamma \geq \beta \)).

(2) A structure \( A = \langle A; F, R \rangle \) is atomic compact if every infinite system of atomic formulas with constants in \( A \) and variables \( x_\gamma \), \( \gamma < \beta \), is simultaneously satisfiable in \( A \) provided the conjunction of every finite subset is satisfiable.

This appendix is concerned with equational compactness. Thus, we will adhere strictly to finitary universal algebras, although many or even most of the results presented can be given a broader setting within the framework of structures or of infinitary structures. In spite of this limitation we introduced atomic compactness to facilitate the presentation of some interesting algebraic constructions of W. Taylor (see §87).

**Remark.** An algebra \( A = \langle A; F \rangle \) is equationally compact if and only if the associated relational system is atomic compact.

Now some examples: Finite algebras (structures) are, of course, equationally (atomic) compact. If an algebra (structure) carries a Hausdorff topology that makes the operations continuous (and has closed relations), then we speak of a topological algebra (topological structure). The next result was observed by J. Mycielski [3].

**Theorem 1.** Compact topological algebras (structures) are equationally compact (atomic compact).