Chapter 6
Probability Theory and Statistics

One might perhaps say that this very thing is probable, that many things happen to men that are not probable.

Agathon, 445 B.C.
(Quoted by Aristotle in the Art of Rhetoric Chapter 24: 1402a)

In this chapter we will review some basic Probability Theory and Statistics to the level that applies to speaker recognition. This coverage is by no means complete. For a more complete treatment of these subjects, the avid reader is referred to [27, 37, 39, 42, 31, 22].

To attain a good understanding of Probability Theory, some basic Measure Theory is presented with coverage of elementary set theory. This framework is then used to define a probability measure and its related concepts as a special case.

6.1 Set Theory

To define the basics of measure theory, some pre-requisite definitions in set theory are given. Let us begin with the definition of a Sample Space.

Definition 6.1 (Sample Space). The Sample Space is the set $\mathcal{X}$ of all possible values that a random variable, $X$, may assume. It is the set of all possible outcomes of an experiment. In other words, it is the domain of the probability function.

Definition 6.2 (Event). An Event $\varepsilon$ is any subset of outputs of an experiment involving a random variable. Namely, $\varepsilon \subset \mathcal{X}$.

Before defining the concept of a measure, consider Figures 6.1, 6.2, 6.3, and 6.4. These figures are Venn diagram representations of different scenarios involving one or two subsets ($\mathcal{A}, \mathcal{B}$). In the figures, $\mathcal{X}$ is the superset, which in measure theory, is known as the sample space. Each of the sets $\mathcal{A}$ and $\mathcal{B}$ may represents an event. $\mathcal{A}$ and $\mathcal{B}$ are called measurable subsets of $\mathcal{X}$ – see Definition 6.20 and its follow-up notes on the completion of a $\sigma$-field. Figure 6.1 corresponds to mutually exclusive or disjoint events. In figure 6.2, the two events $\mathcal{A}$ and $\mathcal{B}$ have a common space. This intersection ($\mathcal{A} \cap \mathcal{B}$) relates to a Logical And, namely, ($\mathcal{A} \wedge \mathcal{B}$) in the sense that an outcome is a member of both sets, $\mathcal{A}$ and $\mathcal{B}$. In some circles, this relationship may
be denoted as \((\mathcal{A}, \mathcal{B})\) [31] and some others use, \((\mathcal{A} \mathcal{B})\) [42]. In this book, we will be using \((\mathcal{A} \cap \mathcal{B})\) to refer to sets and \((\mathcal{A}, \mathcal{B})\) or \((\mathcal{A} \cap \mathcal{B})\), interchangeably, when referring to events. Figure 6.3 indicates the union of two events, \((\mathcal{A} \cup \mathcal{B})\). It is related to the Logical Or, namely, \((\mathcal{A} \lor \mathcal{B})\), and is sometimes written as \((\mathcal{A} + \mathcal{B})\) [42]. The case described by figure 6.4 refers to any outcome which is not a part of event \(\mathcal{A}\). It is denoted as \(\mathcal{A}^c\) in this text. Also, whenever there is ambiguity about the universe, then the complement is denoted in terms of the difference between the universe, \(\mathcal{X}\) and the set \(\mathcal{A}\) written by the following notation, \(\mathcal{X} \setminus \mathcal{A}\), which may be viewed as the complement of \(\mathcal{A}\) with respect to the universe, \(\mathcal{X}\). It is related to the Logical Not and in the notation of logic it may be written as \((\neg \mathcal{A})\). It is sometimes written as \(\mathcal{A}'\) or \(\neg \mathcal{A}\).

Now let us examine the very intuitive but powerful, De Morgan’s law.

**Law 6.1 (De Morgan’s Law).** De Morgan’s law states the following two identities,

\[
\left\{ \bigcup_{i=1}^{N} \mathcal{A}_i \right\}^c = \bigcap_{i=1}^{N} \mathcal{A}_i^c
\]  

(6.1)