In our discussion of the accidental degeneracy of the hydrogen atom in Section 10.2 it was noted that this degeneracy is the consequence of an additional symmetry of the system, a symmetry beyond that associated with a central potential. This additional symmetry manifests itself classically as an additional constant of the motion and quantum mechanically as another operator that commutes with the Hamiltonian. In this chapter we will discuss the properties of this operator and its effect on the energy eigenfunctions of the TISE. We will also discuss its relationship to angular momentum. First, however, we will synopsize the classical Kepler problem.

11.1 The Classical Kepler Problem

The mathematical description of planetary orbits is called the Kepler problem because it was Kepler who deduced, purely empirically, that the planets travel around the sun in elliptical orbits. Newton solved the problem mathematically by ignoring the other planets so his solution of the two-body problem is analogous to that of the classical hydrogen atom. When a particle is subject to a central force its motion is confined to a plane because the angular momentum $L$ is conserved. Moreover, if the particle is bound, this planar motion is confined between two values of $r$. The motion is not necessarily periodic inasmuch as it may never retrace itself. If, however, the force is an attractive inverse square force, a $1/r$ potential, then the bound motion is periodic and the particle executes a closed elliptical orbit. The uniqueness of a classical Keplerian orbit is illustrated in Fig. 11.1.

In Fig. 11.1(a), the potential is an attractive $1/r$ potential while in Fig. 11.1(b) a small non-Keplerian term has been added to the $1/r$ potential. The effect of the added term is to cause a precession of the ellipse so that, except in special circumstances, the particle trajectory never retraces itself. Thus, if the plot of the precessing ellipses were permitted to run for a very long time, the annular region between the minimum and maximum distances from the force center would be completely black.

An additional constant of the motion is responsible for the special properties of the Kepler problem. This constant is a vector that goes by a variety of names [1][2]. For simplicity, we refer to it as the Lenz vector. Consider first a general central
potential $U(r)$ such that the force is $f(r)\hat{a}_r$ where $\hat{a}_r = r/r$ is the unit direction in the $r$ direction[3]. Newton’s second law may be written in terms of the linear momentum $\mathbf{p}$:

$$\dot{\mathbf{p}} = f(r)\hat{a}_r \quad (11.1)$$

Taking the cross product $\dot{\mathbf{p}} \times \mathbf{L}$ where $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}}$ we have

$$\dot{\mathbf{p}} \times \mathbf{L} = m f(r) \left[ \frac{\mathbf{r}}{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) \right]$$

$$= m f(r) r^2 \left[ \frac{\dot{\mathbf{r}} r - \dot{r} \hat{r}}{r^2} \right] \quad (11.2)$$

where we have used the vector identity for the triple cross product as well as the relation

$$\dot{\mathbf{r}} \cdot \mathbf{r} = \frac{1}{2} \frac{d}{dt} (\mathbf{r} \cdot \mathbf{r}) = \dot{r} r \quad (11.3)$$

to arrive at Equation 11.2. Because we are dealing with a central force, $\mathbf{L}$ is a constant so we may rewrite Equation 11.2 as

$$\frac{d}{dt} (\mathbf{p} \times \mathbf{L}) = -m f(r) r^2 \frac{d\hat{a}_r}{dt} \quad (11.4)$$

Inserting Coulomb’s law into Equation 11.4 we find that the additional constant of the motion, the Lenz vector $\mathbf{A}$, is:

$$\mathbf{A} = \left[ (\mathbf{p} \times \mathbf{L}) - \left( \frac{me^2}{4\pi \varepsilon_0} \right) \hat{a}_r \right]$$

$$= \mathbf{p} \times \mathbf{L} - \hat{a}_r \quad \text{a.u.} \quad (11.5)$$

As defined in Equation 11.5, the vector $\mathbf{A}$ lies in the plane of the orbit and points along the major axis of the ellipse toward the perigee. Because it is in the plane of the orbit, it is perpendicular to the angular momentum so that