Chapter 5
Quasi–Monte Carlo Constructions

5.1 Introduction

In this chapter and the following one, we discuss the use of low-discrepancy sampling to replace the pure random sampling that forms the backbone of the Monte Carlo method. Using this alternative sampling method in the context of multivariate integration is usually referred to as quasi–Monte Carlo. A low-discrepancy sample is one whose points are distributed in a way that approximates the uniform distribution as closely as possible. Unlike for random sampling, points are not required to be independent. In fact, the sample might be completely deterministic.

Any attempt to construct such samples requires a precise way of measuring their “uniformity”, so that we can compare different constructions and also make sure that we are indeed improving on random sampling. In fact, we are already familiar with the idea of measuring the uniformity of a point set from our discussion in Sect. 3.5 on theoretical tests for random number generators. Recall that there we were looking at the $s$-dimensional set $\Psi_s$ representing all possible sequences of $s$ successive numbers that can be produced by the generator, and our goal was to make sure this set was “as uniform as possible”. We saw that sets $\Psi_s$ arising from MRGs had a lattice structure that could be assessed via the spectral test, whereas $\mathbb{F}_2$-linear generators were producing sets $\Psi_s$ whose uniformity could be measured via the concept of equidistribution through the resolution and $t$-value. As we will see later in this chapter, these uniformity measures can also be used for assessing the quality of low-discrepancy samples designed for quasi–Monte Carlo. But we will also see that many other measures can be used for that purpose.

As a first step, let us introduce a way of measuring the uniformity of a point set that is not specific to a particular type of construction. More precisely, the idea is to measure the distance between the empirical distribution induced by the point set and the uniform distribution via the Kolmogorov-Smirnov statistic. The concept of discrepancy, which is heavily used in the
quasi–Monte Carlo community — among other things in the terminology low-
discrepancy point set/sequence — looks precisely at such distance measures.
To present these ideas, let us first consider the one-dimensional case.

Consider samples \( P_n \) of size \( n \) over the unit interval \([0, 1)\). An obvious
choice for a low-discrepancy sample \( P_n \) is \( \{0, 1/n, 2/n, \ldots, (n - 1)/n\} \), or
maybe \( \{1/2n, 3/2n, \ldots, (2n - 1)/2n\} \). Alternatively to these two deterministic
choices, one could also use a randomized version,

\[
P_n(v) := \{v \mod 1, (1/n + v) \mod 1, \ldots, ((n - 1)/n + v) \mod 1\},
\]

where \( v \sim U(0, 1) \). The higher uniformity of these one-dimensional samples
can be stated in various ways that more or less all relate to the fact that the
distance between adjacent pairs of points in those samples is equal to \( 1/n \). As
a consequence, if we look at the empirical CDF induced by these samples, it
is always within \( 1/n \) of the CDF of the uniform distribution over \([0, 1)\). That
is, consider the quantity

\[
D^*(P_n) = \sup_{x \in [0, 1)} |F(x) - \hat{F}_n(x)|,
\]

where for \( 0 \leq x < 1 \), \( F(x) = x \) is the CDF of a \( U(0, 1) \) random variable and
\( \hat{F}_n(x) \) is the empirical CDF induced by \( P_n \). That is,

\[
\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{u_i \leq x},
\]

which is the proportion of the numbers \( u_i \) that are smaller than or equal to \( x \).
Then we have that

\[
D^*(\{0, 1/n, 2/n, \ldots, (n - 1)/n\}) = 1/n,
\]

\[
D^*(\{1/2n, 3/2n, \ldots, (2n - 1)/2n\}) = 1/2n,
\]

and

\[
D^*(P_n(v)) = \max\left(v - \frac{|nv|}{n}, \frac{(|nv| + 1)}{n} - v\right) \leq \frac{1}{n}.
\]

We illustrate in Fig. 5.1 how, for the point set \( \{1/2n, 3/2n, \ldots, (2n - 1)/2n\} \)
with \( n = 5 \), the distance between \( \hat{F}_n(x) \) and \( F(x) \) is never more than \( 1/2n \).

Comparing this with a truly random sample \( P_n \), we see that if we are
unlucky, \( D^*(P_n) \) could be much larger than \( 1/n \) in that case. For instance,
for given integers \( k \in \{1, \ldots, n - 1\} \) and \( j \in \{0, \ldots, n - k\} \), with probability
\((n-k)/n)^n\), a given interval of the form \([j/n, (j+k)/n]\) will contain no point,
hence creating a difference of at least \( k/2n \) with the uniform distribution.

Looking at the one-dimensional case helps give an idea of what low-
discrepancy sampling is and how it differs from random sampling. However,
the real challenge arises in the multidimensional case, where we need to find
a way of improving on random sampling without resorting to grids of the