The Partition Functions

The main purpose of this chapter is to discuss the theory of Dahmen–Micchelli describing the difference equations that are satisfied by the quasipolynomials that describe the partition function $T_X$ on the big cells. These equations allow also us to develop possible recursive algorithms.

Most of this chapter follows very closely the paper we wrote with M. Vergne [44]; see also [45].

13.1 Combinatorial Theory

As usual, in this chapter $\Lambda$ is a lattice in a real vector space $V$ and $X$ a list of vectors in $\Lambda$.

13.1.1 Cut-Locus and Chambers

In this section we assume that $X$ generates a pointed cone $C(X)$ and recall some definitions from Section 1.3.3. The set of singular points $C^{\text{sing}}(X)$ is defined as the union of all the cones $C(Y)$ for all the subsets of $X$ that do not span the space $V$.

The set of strongly regular points $C^{\text{reg}}(X)$ is the complement of $C^{\text{sing}}(X)$. The big cells are the connected components of $C^{\text{reg}}(X)$. The big cells are the natural open regions over which the multivariate spline coincides with a polynomial (Theorem 9.7).

Let us recall some properties of the cut locus (Definition 1.54). We have seen in Proposition 1.55 that the translates $C^{\text{sing}}(X) + \Lambda$ give a periodic hyperplane arrangement. The union of the hyperplanes of this periodic arrangement is the cut locus, and each connected component of its complement is a chamber. Each chamber of this arrangement, by Theorem 2.7, is the interior of a bounded polytope. The function $B_X$ is a polynomial on each chamber. The set of chambers is invariant under translation by elements of $\Lambda$. If $X$ is a basis of the lattice $\Lambda$, we have that the chambers are open parallelepipeds.
We introduce a very convenient notation.

**Definition 13.1.** Given two sets $A, B$ in $V$, we set
\[
\delta(A \mid B) := \{ \alpha \in A \mid (A - \alpha) \cap B \neq \emptyset \} = (A - B) \cap \Lambda. \tag{13.1}
\]

Notice that given $\beta, \gamma \in \Lambda$, we have that
\[
\delta(A \mid B) + \beta - \gamma = \delta(A + \beta \mid B + \gamma).
\]

We shall apply this definition mostly to zonotopes and so usually write $\delta(A \mid X)$ instead of $\delta(A \mid B)$.

**Remark 13.2.** If $r$ is a regular vector (i.e., a vector that does not lie in the cut locus), the set $\delta(r \mid X)$ depends only on the chamber $c$ in which $r$ lies and equals $\delta(c \mid X)$.

That is, $\gamma \in \delta(c \mid X)$ if and only if $c \subset \gamma + B(X)$.

The following proposition is an immediate generalization of Proposition 2.50 (which depends on the decomposition of the zonotope into parallelepipeds).

**Proposition 13.3.** If $r$ is a regular vector, then $\delta(r \mid X)$ has cardinality $\delta(X)$. Furthermore, if we consider the sublattice $\Lambda_X$ of $\Lambda$ generated by $X$, every coset with respect to $\Lambda$ meets $\delta(r \mid X)$.

It is often useful to perform a reduction to the nondegenerate case (see Definition 2.23). If $X$ spans $\Lambda$ and $X = X_1 \cup X_2$ is a decomposition, we have that $\Lambda = \Lambda_1 \oplus \Lambda_2$, where $\Lambda_1, \Lambda_2$ are the two lattices spanned by $X_1, X_2$. We have then $C(X) = C(X_1) \times C(X_2)$, $B(X) = B(X_1) \times B(X_2)$, and the big cells for $X$ are products $\Omega_1 \times \Omega_2$ of big cells for the two subsets. The same is also true for the chambers. If we normalize the volumes so that $\Lambda, \Lambda_1, \Lambda_2$ all have covolume 1 we have
\[
\delta(X) = \delta(X_1)\delta(X_2) = \text{vol}(B(X_1))\text{vol}(B(X_2)) = \text{vol}(B(X)).
\]

Finally, for a chamber $c_1 \times c_2$ we have
\[
\delta(c_1 \times c_2 \mid X) = \delta(c_1 \mid X_1) \times \delta(c_2 \mid X_2).
\]

These formulas allow us to restrict most of our proofs to the case in which $X$ is nondegenerate.

### 13.1.2 Combinatorial Wall Crossing

Here we are going to assume that $X$ generates $V$ and use the notation of Section 13.1.1. Let us recall a few facts from Section 2.1.2. Given a chamber, its closure is a compact convex polytope and we have a decomposition of the entire space into faces. We shall use fraktur letters as $\mathfrak{f}$ for these faces.