In this chapter we want to give a taste to the reader of the wide area of approximation theory. This is a very large subject, ranging from analytical to even engineering-oriented topics. We merely point out a few facts more closely related to our main treatment. We refer to [70] for a review of these topics.

We start by resuming and expanding the ideas and definitions already given in Chapter 6.

18.1 Approximation Theory

As usual, we take an $s$-dimensional real vector space $V$ in which we fix a Euclidean structure, and denote by $dx$ the corresponding Lebesgue measure. We also fix a lattice $\Lambda \subset V$ and a list $X$ of vectors in $\Lambda$ spanning $V$ as a vector space and generating a pointed cone.

18.1.1 Scaling

We use the notation of Section 17.1.2. Corollary 17.8 tells us that the space $D(X)$ coincides with the space of polynomials in the cardinal spline space $S_X$. This has a useful application for approximation theory. In order to state the results, we need to introduce some notation.

For every positive real number $h$ we have the scale operator (see (6.8))

$$(\sigma_h f)(x) := f(x/h).$$

In particular, we shall apply this when $h = n^{-1}$, $n \in \mathbb{N}^+$, so that $h\Lambda \supset \Lambda$ is a refinement of $\Lambda$.

Recall that in Proposition 6.5 we have seen that the space $\sigma_h(S_X)$ equals the cardinal space $S_{hX}$ with respect to the lattice $h\Lambda$. 
Remark 18.1. If $f$ is supported in a set $C$, then $\sigma_h f$ is supported in $hC$.

If $U$ is a domain, we have

$$\int_U f \, dx = h^{-s} \int_{hU} \sigma_h f \, dx.$$  

Corollary 18.2. For box splines we have

$$\sigma_h B_X = h^s B_{hX}$$

Proof. By definition,

$$\int_V B_X(x)f(x) \, dx = \int_0^1 \cdots \int_0^1 f \left( \sum_{j=1}^m t_j a_j \right) dt_1 \cdots dt_m.$$

Thus

$$\int_V \sigma_h(B_X(x)f(x)) \, dx = h^s \int_0^1 \cdots \int_0^1 f \left( \sum_{j=1}^m t_j a_j \right) dt_1 \cdots dt_m$$

$$= h^s \int_0^1 \cdots \int_0^1 (\sigma_h f) \left( \sum_{j=1}^m t_j ha_j \right) dt_1 \cdots dt_m$$

$$= \int_V B_{hX}(x)(\sigma_h f)(x) \, dx.$$

So the claim follows from the definition.

We define the scaling operator on distributions by duality. So on a test function $f$, 

$$\langle \sigma_h(T) | f \rangle := \langle T | \sigma_h^{-1}(f) \rangle = \langle T | \sigma_{1/h}(f) \rangle.$$  (18.1)  

In particular, $\sigma_h \delta_a = \langle \delta_a | f(hx) \rangle = f(ha)$, so that 

$$\sigma_h \delta_a = \delta_{ha}, \quad \sigma_h \sum_a f(a) \delta_a = \sum_a f(a/h) \delta_a.$$  

Observe that $\sigma_h$ acts as an automorphism with respect to convolution.

Notice that if $T$ is represented by a function $g$, i.e., $\langle T | f \rangle = \int_V g(x)f(x) \, dx$, we have 

$$\langle \sigma_h(T) | f \rangle = \int_V g(x)f(hx) \, dx = h^{-s} \int_V g(h^{-1}x)f(x) \, dx,$$

and thus $\sigma_h(T)$ is represented by the function $h^{-s} \sigma_h(g)$ and not $\sigma_h(g)$.

The relation between scaling and the Laplace transform is 

$$L(\sigma_h f) = h^s \sigma_{1/h} L(f).$$  (18.2)