The equations of motion of single- and multi-degrees-of-freedom undamped vibrating systems are obtained both by writing the dynamic equilibrium equations and by resorting to Lagrange equations. The vibrating system is assumed to be constrained to either an inertial reference frame or a body moving with a known time history with respect to an inertial frame and whose motion is translational. The equations of motion are obtained both in the configuration and in the state space.

1.1 Oscillator with a single degree of freedom

The simplest system studied by structural dynamics is the linear mechanical oscillator with a single degree of freedom. It consists of a point mass suspended by a massless linear spring (Fig. 1.1a). Historically, however, the mathematical pendulum (Fig. 1.1c) represented for centuries the most common paradigm of an oscillator, which could be assumed to be linear, at least within adequate limitations.

The study of the simple linear oscillators of Fig. 1.1 is important for more than just historical reasons. In the first instance it is customary to start the study of mechanics of vibration with a model that is very simple but demonstrates, at least qualitatively, the behavior of more complex systems.

The arrangements shown in Fig. 1.1 also have a great practical importance: They constitute models that can often be used to study, with good approximation, the behavior of systems of greater complexity. Moreover,
FIGURE 1.1. Linear oscillators with one degree of freedom: (a) Spring–mass system; the coordinate $x$ for the study of the motion of point P can expressed in an inertial reference frame or be a relative displacement; (b) physical pendulum; (c) mathematical pendulum.

systems with many degrees of freedom, and even continuous systems, can be reduced, under fairly wide simplifying assumptions, to a set of independent systems with a single degree of freedom.

A linear spring is an element that, when stretched of the quantity $l - l_0$, reacts with a force

$$F_s = -k(l - l_0),$$

(1.1)

where $l_0$ is the length at rest of the spring and $k$ is a constant, usually referred to as the stiffness of the spring, expressing the ratio between the force and the elongation. In SI units, it is measured in N/m. If constant $k$ is positive, the force is a restoring force, opposing the displacement of point P. The system is then statically stable, in the sense that, when displaced from its equilibrium position, it tends to return to it.\(^1\)

A force function of time $F(t)$ can act on point P and the supporting point A can move in $x$-direction with a known time history $x_A(t)$.

The dynamic equilibrium equation states that the inertia force is, at any time, in equilibrium with the elastic reaction of the spring added to the external forces. Written with reference to the inertial $x$-coordinate, it is simply

$$m\ddot{x} = -k[x - l_0 - x_A(t)] + F(t) - mg.$$  

(1.2)

Owing to the linearity of the system of Fig. 1.1a, the length at rest of the spring $l_0$ and all constant forces such as those due to the gravitational acceleration $g$, affect the static equilibrium position but not its dynamic behavior. The dynamic problem can thus be separated from the

\(^1\)For a more detailed definition of stability see Chapter 20. Only stable systems will be dealt with in this chapter.