Chapter 3
Self-dual Lagrangians on Phase Space

At the heart of this theory is the interplay between certain automorphisms and Legendre transforms. The main idea originates from the fact that a large class of PDEs and evolution equations—the completely self-dual differential systems—can be written in the form

\[(p,x) \in \partial L(x,p),\]

where \(\partial L\) is the subdifferential of a self-dual Lagrangian \(L : X \times X^* \to \mathbb{R} \cup \{+\infty\}\) on phase space. This class of Lagrangians is introduced in this chapter, where its remarkable permanence properties are also established, in particular, their stability under various operations such as convolution, direct sum, superposition, iteration, and certain regularizations, as well as their composition with skew-adjoint operators.

3.1 Invariance under Legendre transforms up to an automorphism

Definition 3.1. Given a bounded linear operator \(R\) from a reflexive Banach space \(E\) into its dual \(E^*\), we say that a convex lower semicontinuous functional \(\ell : E \to \mathbb{R} \cup \{+\infty\}\) is \(R\)-self-dual if

\[\ell^*(Rx) = \ell(x)\]

for any \(x \in E\), \(\ell^*\) is the Legendre transform of \(\ell\) on \(E\).

The following easy proposition summarizes the properties of \(R\)-self-dual functions to be used in what follows.

Proposition 3.1. Let \(\ell\) be an \(R\)-self-dual convex functional on a reflexive Banach space \(E\), where \(R : E \to E^*\) is a bounded linear operator. Then,

1. For every \(x \in E\), we have \(\ell(x) \geq \frac{1}{2} \langle Rx, x \rangle\).
2. For \(\bar{x} \in E\), we have \(\ell(\bar{x}) = \frac{1}{2} \langle R\bar{x}, \bar{x} \rangle\) if and only if \(R\bar{x} \in \partial \ell(\bar{x})\).

Proof. It is sufficient to combine self-duality with the Fenchel-Legendre inequality to obtain

\[2\ell(x) = \ell^*(Rx) + \ell(x) \geq \langle Rx, x \rangle \] with equality if and only if \(Rx \in \partial \ell(x)\).

This leads us to introduce the following definition.

**Definition 3.2.** The \(R\)-core of \(\ell\) is the set

\[\mathcal{C}_R \ell = \{x \in E; Rx \in \partial \ell(x)\} = (R - \partial \ell)^{-1}\{0\}.
\]

It is easy to see that the only functional satisfying \(\varphi^*(x) = \varphi(x)\) (i.e., when \(R\) is the identity) is the quadratic function \(\varphi(x) = \tfrac{1}{2}\|x\|^2\). In this case, the \(I\)-core of \(\varphi\) is the whole space. On the other hand, by simply considering the operator \(Rx = -x\), we can see that the notion becomes much more interesting. The following proposition is quite easy to prove.

**Proposition 3.2.** Let \(E\) be a reflexive Banach space.

1. If \(R\) is self-adjoint and satisfies \(\langle Rx, x \rangle \geq \delta \|x\|^2\) for some \(\delta > 0\), then the only \(R\)-self-dual function on \(E\) is \(\ell(x) = \frac{1}{2} \langle Rx, x \rangle\). In this case, \(\mathcal{C}_R \ell = E\).

2. On the other hand, for every \(a\) in a Hilbert space \(H\), the function

\[\ell_a(x) = \frac{1}{2}\|x\|^2 - 2\langle a, x \rangle + \|a\|^2\]

satisfies \(\ell_a^*(-x) = \ell_a(x)\) for every \(x \in H\). In this case, \(\mathcal{C}_{-I} \ell = \{a\}\).

3. If \(E = E_1 \times E_2\) is a product space and \(R(x_1, x_2) = (R_1 x_1, R_2 x_2)\), where \(R_i : E_i \to E_i^*\) \((i = 1, 2)\), then a function on \(E_1 \times E_2\) of the form \(\ell(x_1, x_2) = \ell_1(x_1) + \ell_2(x_2)\) is \(R\)-self-dual as long as \(\ell_1\) is \(R_1\)-self-dual and \(\ell_2\) is \(R_2\)-self-dual. In this case, \(\mathcal{C}_R \ell = \mathcal{C}_{R_1} \ell_1 \times \mathcal{C}_{R_2} \ell_2\). In particular, for any \(a\) in a Hilbert space \(E_2\), the function

\[\ell(x_1, x_2) = \frac{1}{2}\|x_1\|^2 + \frac{1}{2}\|x_2\|^2 - 2\langle a, x_2 \rangle + \|a\|^2\]

is \((I, -I)\)-self-dual on \(E_1 \times E_2\) and \(\mathcal{C}_{(I, -I)} \ell = E_1 \times \{a\}\).

4. If \(R(x_1, x_2) = (x_2, x_1)\) and \(S(x_1, x_2) = (-x_2, -x_1)\) from a Hilbert space \(H \times H\) into itself, then for any convex lower semicontinuous functions \(\psi\) on \(H\) and any skew-adjoint operator \(A : H \to H\), the function defined for \((x_1, x_2) \in E = H^2\) by

\[\ell_1(x_1, x_2) = \psi(x_1) + \psi^*(Ax_1 + x_2)\quad (\text{resp., } \ell_2(x_1, x_2) = \psi(x_1) + \psi^*(-Ax_1 - x_2))\]

is \(R\)-self-dual (resp., \(S\)-self-dual) on \(E = H^2\). In this case,

\[\mathcal{C}_R \ell_1 = \{(x_1, x_2) \in H \times H; x_2 = -Ax_1 + \partial \psi(x_1)\}\]

and

\[\mathcal{C}_S \ell_2 = \{(x_1, x_2) \in H \times H; x_2 = -Ax_1 - \partial \psi(x_1)\}\].