Chapter 5
The Nonlinear Schrödinger Equation. Local Theory

In this chapter we shall study of local well-posedness of the nonlinear IVP,

\[
\begin{aligned}
    i\partial_t u & = -\Delta u - \lambda |u|^\alpha u, \\
    u(x,0) & = u_0(x),
\end{aligned}
\tag{5.1}
\]

\(t \in \mathbb{R},\ x \in \mathbb{R}^n\), where \(\lambda\) and \(\alpha\) are real constants with \(\alpha > 1\).

The equation in (5.1) appears as model in several physical problems (see references [GV1], [N], [SCMc], [ZS]).

Formally solutions of problem (5.1) satisfy the following conservation laws, that is, if \(u(x,t)\) is solution of (5.1) then for all \(t \in [0,T]\), the \(L^2\)-norm

\[
\|u(\cdot,t)\|_2 = \|u_0\|_2,
\tag{5.2}
\]

the energy

\[
\int_{\mathbb{R}^n} (|\nabla_x u(x,t)|^2 - \frac{2\lambda}{\alpha+1} |u(x,t)|^{\alpha+1}) \, dx = \|\nabla u_0\|_2^2 - \frac{2\lambda}{\alpha+1} \|u_0\|_2^{\alpha+1},
\tag{5.3}
\]

the momentum

\[
\mathcal{J} \int_{\mathbb{R}^n} \nabla u(x,t) \bar{u}(x,t) \, dx = \mathcal{J} \int_{\mathbb{R}^n} \nabla u_0(x) \bar{u}_0(x) \, dx,
\tag{5.4}
\]

and the so-called quasiconformal law [GV1]

\[
\| (x+2it \nabla)u(t) \|_2^2 - \frac{8\lambda t^2}{\alpha+1} \|u(t)\|_2^{\alpha+1} \|u(t)\|_2^{\alpha+1} \\
= \|xu_0\|_2^2 - 4\lambda \left( \frac{4n(\alpha-1)}{\alpha+1} \right) \int_0^t \left( \int_{\mathbb{R}^n} |u(x,s)|^{\alpha+1} \, dx \right) \, ds.
\tag{5.5}
\]

We will use these identities in the next chapter.

We shall say that the equation in (5.1) is focusing if \( \lambda > 0 \) (attractive nonlinearity) and defocusing if \( \lambda < 0 \) (repulsive nonlinearity).

In any dimension the equation in (5.1) in the focusing case \( \lambda > 0 \) has solutions of the form,

\[
    u(x, t) = e^{i\omega t} \varphi(x),
\]

called standing waves or ground states, which are closely related to the elliptic problem

\[
    -\Delta u = f(u),
\]

which have been extensively studied. In our case, \( f(u) = -\omega u + |u|^{\alpha-1}u \), with \( \omega > 0 \) (otherwise there is no solution for (5.7)) and \( \lambda = 1 \). Indeed, the problem is to find \( \varphi \in H^1(\mathbb{R}^n) \) such that

\[
    -\Delta \varphi + \omega \varphi = |\varphi|^{\alpha-1} \varphi.
\]

The existence of solutions of the equation (5.8) in dimension \( n \geq 3 \) was established by Strauss [Sr2] and Berestycki and Lions [BLi] (see also [BLiP]). The bi-dimensional case was proved in [BGK] by Berestycki, Gallouët, and Kavian. Regarding uniqueness of solutions of (5.8), Kwong [Kw1] showed that positive solutions of the problem (5.7) with \( f(u) = -u + u^p \) are unique up to translations. We summarize these results in the next theorem.

**Theorem 5.1.** Let \( n \geq 2 \) and \( 1 < \alpha < (n+2)/(n-2) \) (1 < \( \alpha < \infty \), \( n = 2 \)). Then there exists a unique positive, spherically symmetric solution of (5.8) \( \varphi \in H^1(\mathbb{R}^n) \). Moreover, \( \varphi \) and its derivatives up to order 2 decay exponentially at infinity.

**Remark 5.1.** The restriction on \( \alpha \) comes from Pohozaev’s identity (5.81) since we want to have \( H^1 \) solutions of (5.8) (see Exercise 5.3).

**Remark 5.2.** There are infinitely many radially symmetric solutions under the hypothesis of Theorem 5.1 without the positivity assumption (see [BLi], [E], [JK]).

As we will see below once we have a solution of (5.1) we can use the invariance of the equation to generate other solutions. Thus if \( u = u(x, t) \) is a solution of the equation (5.1) then the following are also solutions:

(i) \( u_\mu(x, t) = \mu^{\frac{n-2}{4}} u(\mu x, \mu^2 t), \mu \in \mathbb{R}, \) with initial data given by \( u_{0\mu}(x) = \mu^{\frac{n-2}{4}} u_0(\mu x) \).

(ii) \( u_\theta(x, t) = e^{i\theta} u(x, t), \theta \in \mathbb{R} \).