Chapter 11

SL$_2$($\mathbb{R}$)

The group SL$_2$($\mathbb{R}$) is the simplest case of a so called reductive Lie group. Harmonic analysis on these groups turns out to be more complex than the previous cases of abelian, compact, or nilpotent groups. On the other hand, the applications are more rewarding. For example, via the theory of automorphic forms, in particular the Langlands program, harmonic analysis on reductive groups has become vital for number theory. In this chapter we prove an explicit Plancherel Theorem for functions in the Hecke algebra of the group $G = \text{SL}_2(\mathbb{R})$. We apply the trace formula to a uniform lattice and as an application derive the analytic continuation of the Selberg zeta function.

11.1 The Upper Half Plane

Let $G = \text{SL}_2(\mathbb{R})$ denote the special linear group of degree 2, i.e.

$$\text{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_2(\mathbb{R}) : ad - bc = 1 \right\}.$$ 

The locally compact group SL$_2$($\mathbb{R}$) acts on the upper half plane

$$\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$$

by linear fractionals, i.e., for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$ and for $z \in \mathbb{H}$ one defines

$$gz = \frac{az + b}{cz + d}.$$
To see that this is well-defined one has to show that $cz + d \neq 0$. If $c = 0$ then $d \neq 0$ and so the claim follows. If $c \neq 0$ then the imaginary part equals $\text{Im}(cz + d) = c\text{Im}(z) \neq 0$. Next one has to show that $gz$ lies in $\mathbb{H}$ if $z$ does and that $(gh)z = g(hz)$ for $g, h \in \text{SL}_2(\mathbb{R})$. The latter is an easy computation, for the former we will now derive an explicit formula for the imaginary part of $gz$. Multiplying numerator and denominator by $c\bar{z} + d$ one gets

$$gz = \frac{ac|z|^2 + bd + 2bc\Re(z) + z}{|cz + d|^2},$$

so in particular,

$$\text{Im}(gz) = \frac{\text{Im}(z)}{|cz + d|^2},$$

which is strictly positive if $\text{Im}(z)$ is. Note that the action of the central element $-1 \in \text{SL}_2(\mathbb{R})$ is trivial.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ stabilizes the point $i \in \mathbb{H}$, then $\frac{ai+b}{ai+d} = i$, or $ai + b = -c + di$, which implies $a = d$ and $b = -c$. So the stabilizer of the point $i \in \mathbb{H}$ is the rotation group:

$$K = \text{SO}(2) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R}, \ a^2 + b^2 = 1 \right\},$$

which also can be described as the group of all matrices of the form

$$\begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \quad \text{for} \quad \varphi \in \mathbb{R}.$$  

The operation of $G$ on $\mathbb{H}$ is transitive, as for $z = x + iy \in \mathbb{H}$ one has

$$z = \begin{pmatrix} \sqrt{y} & x \\ \sqrt{y} & \frac{1}{\sqrt{y}} \end{pmatrix} i.$$

It follows that via the map

$$G/K \rightarrow \mathbb{H}$$

$$gK \mapsto gi,$$

the upper half plane $\mathbb{H}$ can be identified with the quotient $G/K$.

**Theorem 11.1.1 (Iwasawa Decomposition)**

Let $A$ be the group of all diagonal matrices in $G$ with positive entries.