Chapter 3
Lagrange’s Equation of Motion

This chapter develops Lagrange’s equation of motion for a class of multi-discipline dynamic systems. To derive Lagrange’s equation we utilize some concepts from analytical dynamics, and the first law of thermodynamics. By carrying out the development using the fundamental variables it is clear that the results obtained are applicable to all the engineering disciplines described in Chapter 1.

Section 3.1 introduces the concepts of generalized displacement, virtual displacement and virtual work. In Section 3.2 Lagrange’s equation of motion is derived starting from the first law of thermodynamics. In Section 3.3 various examples are used to illustrate the application of Lagrange’s equation to mechanical, electrical, fluid, and multidiscipline systems.

3.1 Analytical Dynamics

As discussed in Chapter 1 dynamic systems can be described as an assemblage of inductors, capacitors, resistors, constraint elements and sources. Associated with each of these elements is the set of fundamental variables displacement, $q$, flow, $f$, effort, $e$, and momentum, $p$. From Paynter’s diagram (Fig. 1.4) it can be observed that all four variables are not all required to determine the state of the system at any time. In fact only two of the fundamental variables are required to determine the states of the system since, the other two variables can be determined using the system properties and the differential/integral relationship between the variables.

In Lagrangian dynamics the displacement and flow variables are used to describe the system behavior. In the Hamiltonian description of dynamic systems the momentum and displacement variables are used to describe the system. The bond graph and linear graph description of dynamic systems use the effort and flow variables to describe the system behavior.
3.1.1 Generalized variables

The number of displacement and flow variables that can be assigned to a dynamic system can be quite large. However, for each system there is a minimum number of displacement variables that are required to uniquely determine the state or configuration of the system. This minimum set of displacement variables are called the \textit{generalized displacements}. The corresponding set of flow variables are called the \textit{generalized flows}. The number of generalized displacement variables used to describe the system is equal to the number of \textit{degrees of freedom}. (See Section 2.2 and Section 2.3.)

\textbf{Example 3.1.}

Consider the spring, mass damper system shown in Figure 3.1a. Figure 3.1b shows the displacement coordinates assigned to the system components. Assigned to the mass is the displacement, $x_1$. The left and right end of the spring are assigned displacement variables $x_2$, and $x_3$, respectively. The left and right end of the damper are assigned displacement variables $x_4$, and $x_5$, respectively. Since the left end of the spring and damper are fixed, it is clear that $x_2 = x_4 = 0$. Also, since the right ends of the spring and damper are attached to the mass we have $x_5 = x_3 = x_1$. Thus, the system can be described using the single displacement variable $x_1$, and the the corresponding flow variable $v_1$. This system has 1 degree of freedom with $x_1$ being the the generalized displacement and $v_1$ the corresponding generalized flow.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{spring_mass_damper.png}
\caption{Spring-mass-damper}
\end{figure}

\textbf{Example 3.2.}

Figure 3.2a shows a resistor, $R$, an inductor, $L$, and a capacitor, $C$, in series with a voltage source, $v$. Figure 3.2b shows the flow variables assigned to the system components. The current $i_1$ is assigned to the voltage source, $i_2$ to the resistor, $i_3$ to the inductor, and $i_4$ to the capacitor. However, Kirchhoff’s