Chapter 6
Volterra-type Variational Inequalities

The variational inequalities studied in this chapter involve the Volterra operator, (4.42), and therefore are history dependent. The term containing this operator appears as a perturbation of the bilinear form $a$ and it is not involved in the function $j$; even if various other cases may be considered, we made this choice since it is suggested by the structure of variational models that describe the frictional contact of viscoelastic materials with long memory. We consider two types of variational inequalities with a Volterra integral term: the first one is elliptic, and the second one is evolutionary. For both inequalities, we provide existence and uniqueness results. Finally, we study the behavior of the solution with respect to the integral term and with respect to the nondifferentiable function and provide convergence results. The results presented in this chapter will be applied in the study of antiplane frictional contact problems involving viscoelastic materials with long memory. Everywhere in this chapter, $X$ is a real Hilbert space with the inner product $(\cdot, \cdot)_X$, and the norm $\|\cdot\|_X$, and $[0, T]$ denotes the time interval of interest, where $T > 0$.

6.1 Volterra-type Elliptic Variational Inequalities

In this section, we consider the problem of finding $u : [0, T] \to X$ such that

$$a(u(t), v - u(t)) + \left( \int_0^t A(t - s) u(s) ds, v - u(t) \right)_X + j(v) - j(u(t)) \geq (f(t), v - u(t))_X \quad \forall v \in X, \ t \in [0, T],$$

(6.1)

where $a$ is a bilinear form on $X$, $j : X \to (-\infty, \infty]$, $A : [0, T] \to \mathcal{L}(X)$ and $f : [0, T] \to X$. Note that problem (6.1) does not involve the time derivative of the unknown $u$ and, therefore, it represents an elliptic variational inequality; moreover, since this problem involves the Volterra operator (4.42)
with \( Y = X \) and \( y_0 = 0_X \), we refer to it as a **Volterra-type elliptic variational inequality**.

In the study of (6.1) we assume that:

\[
a : X \times X \to \mathbb{R} \text{ is a bilinear symmetric form and}
\begin{align*}
(a) & \text{ there exists } M > 0 \text{ such that } |a(u,v)| \leq M \|u\|_X \|v\|_X \quad \forall u, v \in X. \\
(b) & \text{ there exists } m > 0 \text{ such that } a(v,v) \geq m \|v\|_X^2 \quad \forall v \in X.
\end{align*}
\]

\[
(6.2)
\]

\[
A \in C([0,T]; \mathcal{L}(X)).
\]

\[
(6.3)
\]

\[
j : X \to (-\infty, \infty] \text{ is a proper convex l.s.c. function.}
\]

\[
(6.4)
\]

\[
f \in C([0,T]; X).
\]

\[
(6.5)
\]

The main result of this section is the following.

**Theorem 6.1.** Let \( X \) be a Hilbert space and assume that (6.2)–(6.5) hold. Then there exists a unique solution \( u \) to problem (6.1). Moreover, the solution satisfies \( u \in C([0,T]; X) \).

**Proof.** The proof will be carried out in several steps, and it is based on the results on time-dependent elliptic variational inequalities presented in Section 3.4 and a fixed point argument. The steps of the proof are the following.

(i) Let \( \eta \in C([0,T]; X) \). We use Theorem 3.12 to see that there exists a unique function \( u_\eta \in C([0,T]; X) \) such that

\[
a(u_\eta(t), v - u_\eta(t)) + (\eta(t), v - u_\eta(t))_X + j(v) - j(u_\eta(t)) \geq (f(t), v - u_\eta(t))_X \quad \forall v \in X, \ t \in [0,T].
\]

(6.6)

(ii) Next, we consider the operator \( A : C([0,T]; X) \to C([0,T]; X) \) given by

\[
A\eta(t) = \int_0^t A(t-s) u_\eta(s) \, ds \quad \forall \eta \in C([0,T]; X), \ t \in [0,T]
\]

(6.7)

and we note that by condition (6.3), the integral in (6.7) is well-defined and the operator \( A \) takes values in the space \( C([0,T]; X) \).

We prove now that \( A \) has a unique fixed point \( \eta^* \in C([0,T]; X) \). To this end, consider two elements \( \eta_1, \eta_2 \in C([0,T]; X) \), denote \( u_1 = u_{\eta_1}, u_2 = u_{\eta_2} \), and let \( t \in [0,T] \). Using (6.7), we find that

\[
\|A\eta_1(t) - A\eta_2(t)\|_X \leq \int_0^t \|A(t-s)\|_{\mathcal{L}(X)} \|u_1(s) - u_2(s)\|_X \, ds
\]

and, keeping in mind (6.3), it follows that

\[
\|A\eta_1(t) - A\eta_2(t)\|_X \leq c \int_0^t \|u_1(s) - u_2(s)\|_X \, ds
\]

(6.8)