Chapter 7
Modeling of Contact Processes

In this chapter, we present a general description of mathematical modeling of the processes involved in contact between a deformable body and an obstacle or a foundation. Such kind of processes abound in industry and everyday life and, for this reason, a considerable effort has been made in their modeling, analysis, and numerical simulations. We present the physical setting, the variables that determine the state of the system, the material behavior that is reflected in a constitutive law, the input data, the equation of evolution for the state of the system, and the boundary conditions for the system variables. In particular, we provide a description of the frictional contact conditions, including versions of the Coulomb law of dry friction and its regularizations. Most of the notions and results we present here are standard and can be found in many books on mechanics and, therefore, we skip many of the details. In this chapter, all variables are assumed to have sufficient degree of smoothness consistent with developments they are involved in.

7.1 Physical Setting

A large variety of situations involving contact phenomena can be cast in the general physical setting shown in Figure 7.1 and described in what follows. A deformable body occupies, in the reference configuration, an open bounded connected set $B \subset \mathbb{R}^3$ with boundary $\partial B$. We denote vectors and tensors by boldface letters, such us the position vector $\mathbf{x} = (x_i) \in B \cup \partial B$. Here and below, the indices $i, j, k, l$ run from 1 to 3; an index that follows a comma indicates a derivative with respect to the corresponding component of the spatial variable $\mathbf{x}$, and the summation convention over repeated indices is adopted. We denote by $\mathbb{S}^3$ the space of second-order symmetric tensors on $\mathbb{R}^3$ or, equivalently, the space of symmetric matrices of order 3. The canonical
inner products and the corresponding norms on $\mathbb{R}^3$ and $\mathbb{S}^3$ are

$$ u \cdot v = u_i v_i, \quad \| v \| = (v \cdot v)^{1/2} \quad \forall u, v \in \mathbb{R}^3, $$

$$ \sigma \cdot \tau = \sigma_{ij} \tau_{ij}, \quad \| \tau \| = (\tau \cdot \tau)^{1/2} \quad \forall \sigma, \tau \in \mathbb{S}^3, $$

respectively.

The boundary $\partial B$ is assumed to be composed of three sets $\Gamma_D, \Gamma_F,$ and $\Gamma_C$, with mutually disjoint relatively open sets $\Gamma_D, \Gamma_F$, and $\Gamma_C$. We assume the boundary $\partial B$ is Lipschitz continuous and therefore the unit outward normal vector $\nu$ exists a.e. on $\partial B$. The body is clamped on $\Gamma_D$. Surface tractions of density $f_2$ act on $\Gamma_F$ and volume forces of density $f_0$ act in $B$. The body is in contact on $\Gamma_C$ with an obstacle, the so-called foundation. The force densities $f_0$ and $f_2$ may depend on time. As a result, the mechanical state of the body evolves on the time interval $[0, T]$, where $T > 0$.

We are interested in providing a mathematical model that describes the evolution of the body in this physical setting. To this end, we denote by $\sigma = \sigma(x,t) = (\sigma_{ij}(x,t))$ the stress field and by $u = u(x,t) = (u_i(x,t))$ the displacement field. The functions $u : B \times [0, T] \to \mathbb{R}^3$ and $\sigma : B \times [0, T] \to \mathbb{S}^3$ will play the role of the unknowns of the contact problem. From time to time, we suppress the explicit dependence of the quantities on the spatial variable $x$, or both $x$ and $t$; i.e., when it is convenient to do so, we write $\sigma(t)$ and $u(t)$, or even $\sigma$ and $u$. Also, everywhere below a dot above a variable will represent the derivative with respect to time and, therefore, $\dot{u}$ denotes the velocity field.

We assume that the volume forces and surface tractions change slowly in time so that the inertia of the mechanical system is negligible. In other words, we neglect the acceleration term in the equation of motion and therefore we consider a quasistatic process of contact. Thus, the stress field satisfies the equation of equilibrium

$$ \text{Div} \, \sigma + f_0 = 0 \quad \text{in} \ B \times (0, T) $$

where Div is the divergence operator, that is, $\text{Div} \, \sigma = (\sigma_{ij,j})$, and recall that $\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}$. Since the body is clamped on $\Gamma_D$, we impose the displacement

Fig. 7.1 A deformable body in contact with a foundation.