Chapter 8
Continuum Mechanics: Three Spatial Dimensions

8.1 Introduction

The water in the ocean, the air in the room, and a rubber ball have a common characteristic, they appear to completely occupy their respective domains. What this means is that the material occupies every point in the domain. This observation is the basis of the continuum approximation, and it was used in Section 5.2 to define continuum variables such as density and flux. These variables can be defined as long as the individual nature of the constituent particles are not apparent. So, for example, the continuum approximation cannot be used on the nanometer scale, because atomic radii range from 0.2 to 3.0 nm. It can, however, be used down to the micron level. As an example, at 15°C, and one atmosphere, there are approximately $3 \times 10^7$ molecules in a cubic micron of air. Similarly, for water at room temperature there are approximately $3 \times 10^{10}$ molecules in a cubic micron, and for a metal such as copper there are approximately $10^{11}$ atoms in a cubic micron. Consequently, the averaging on which the continuum approximation is based is applicable down to the micron scale. This is why continuum models are commonly used for microdevices, which involve both electrical and mechanical components. At the other extreme, continuum models are used to study the motion of disk galaxies and, more recently, to investigate the existence and properties of the “dark fluid” proposed to be responsible for the expansion of the universe. This range of applicability is why the continuum approximation, and the subsequent equations of motion, play a fundamental role in most branches of science and engineering. From a mathematical standpoint, the problems that come from continuum models have been almost single-handedly responsible for the development of an area central to applied mathematics, and this is the theory of nonlinear partial differential equations. In this chapter the fundamental concepts of continuum mechanics are introduced, and they are then used to derive equations of motion for viscous fluids and elastic solids.
8.2 Material and Spatial Coordinates

To define the material coordinate system, assume that at \( t = 0 \) a particular point in the material is located at \( \mathbf{x} = \mathbf{A} \). It is assumed that as the material moves, the position of the point is given as \( \mathbf{x} = \mathbf{X}(\mathbf{A}, t) \). To be consistent, the position function must satisfy \( \mathbf{X}(\mathbf{A}, 0) = \mathbf{A} \). The resulting displacement and velocity functions are defined as

\[
\mathbf{U}(\mathbf{A}, t) = \mathbf{X}(\mathbf{A}, t) - \mathbf{A},
\]

and

\[
\mathbf{V}(\mathbf{A}, t) = \frac{\partial \mathbf{U}}{\partial t}.
\]

Because \( \mathbf{X}(\mathbf{A}, 0) = \mathbf{A} \), it follows that \( \mathbf{U}(\mathbf{A}, 0) = \mathbf{0} \).

Instead of following particles as they move, one can select a spatial location and then let them come to you. This is the viewpoint taken for spatial coordinates. In this system, the displacement function is denoted as \( \mathbf{u}(\mathbf{x}, t) \), and the velocity is \( \mathbf{v}(\mathbf{x}, t) \). As is usual for displacement functions, it is required that \( \mathbf{u}(\mathbf{x}, 0) = \mathbf{0} \).

**Example**

Suppose a particle that started at location \((1, -1, 1)\) is, at \( t = 2 \), located at \((3, 0, -1)\).

**Material Coordinates:** For this particle, \( \mathbf{A} = (1, -1, 1) \), and its displacement at \( t = 2 \) is \((3, 0, -1) - (1, -1, 1) = (2, 1, -2)\). In other words,

\[
\mathbf{U}(\mathbf{A}, 2) = (2, 1, -2), \quad \text{for } \mathbf{A} = (1, -1, 1).
\]

We also have that

\[
\mathbf{X}(\mathbf{A}, 0) = (1, -1, 1), \quad \text{for } \mathbf{A} = (1, -1, 1),
\]

and

\[
\mathbf{X}(\mathbf{A}, 2) = (3, 0, -1), \quad \text{for } \mathbf{A} = (1, -1, 1).
\]

**Spatial Coordinates:** At \( t = 2 \), the displacement of the particle located at \((3, 0, -1)\) is \((2, 1, -2)\). In other words,

\[
\mathbf{u}(\mathbf{x}, 2) = (2, 1, -2), \quad \text{for } \mathbf{x} = (3, 0, -1).
\]

The spatial system is the one usually used for fluids, which includes both gases and liquids. As an example, when measuring the properties of the atmosphere, observers are often fixed, and not moving with the air. This is the viewpoint taken by the spatial coordinate system, and hence the reason why