Chapter V

Martingales and Stochastics

This chapter is to introduce the vocabulary for describing the evolution of random systems over time. It will also cover the basic results of classical martingale theory and mention some basic processes such as Markov chains, Poisson processes, and Brownian motion. This chapter should be treated as a reference source for chapters to come.

We start with generalities on filtrations and stopping times, go on to martingales in discrete time, and then to finer results on martingales and filtrations in continuous time. Throughout, \((\Omega, \mathcal{H}, \mathbb{P})\) is a fixed probability space in the background, and all stochastic processes are indexed by some set \(T\), which is either \(\mathbb{N} = \{0, 1, \ldots \}\) or \(\mathbb{R}_+ = [0, \infty)\) or some other subset of \(\bar{\mathbb{R}} = [-\infty, +\infty]\). We think of \(T\) as the time-set; its elements are called times. On a first reading, the reader should take \(T = \mathbb{N}\).

1 Filtrations and Stopping Times

Let \(T\) be a subset of \(\bar{\mathbb{R}}\). A filtration on \(T\) is an increasing family of sub-\(\sigma\)-algebras of \(\mathcal{H}\) indexed by \(T\); that is, \(\mathcal{F} = (\mathcal{F}_t)_{t \in T}\) is a filtration if each \(\mathcal{F}_t\) is a \(\sigma\)-algebra on \(\Omega\), each \(\mathcal{F}_t\) is a subset of \(\mathcal{H}\), and \(\mathcal{F}_s \subset \mathcal{F}_t\) whenever \(s < t\). Given a stochastic process \(X = (X_t)_{t \in T}\), letting \(\mathcal{F}_t = \sigma\{X_s : s \leq t\}\) for each time \(t\), we obtain a filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \in T}\); it is called the filtration generated by \(X\).

Heuristically, we think of a filtration \(\mathcal{F}\) as a flow of information, with \(\mathcal{F}_t\) representing the body of information accumulated by time \(t\) by some observer of the ongoing experiment modeled by \((\Omega, \mathcal{H}, \mathbb{P})\). Or, we may think of \(\mathcal{F}_t\) as the collection of \(\bar{\mathbb{R}}\)-valued random variables \(V\) such that the observer can tell the value \(V(\omega)\) at the latest by time \(t\), whatever the outcome \(\omega\) turns out to be. Of course, it is possible to have different observers with different

E. Çınlar, Probability and Stochastics, Graduate Texts in Mathematics 261, DOI 10.1007/978-0-387-87859-1_5,
© Springer Science+Business Media, LLC 2011
information flows. Given two filtrations $\mathcal{F}$ and $\mathcal{G}$, we say that $\mathcal{F}$ is finer than $\mathcal{G}$, or that $\mathcal{G}$ is coarser than $\mathcal{F}$, if $\mathcal{F}_t \supset \mathcal{G}_t$ for every time $t$.

**Adaptedness**

Let $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$ be a filtration. Let $X = (X_t)_{t \in \mathbb{T}}$ be a stochastic process with some state space $(E, \mathcal{E})$. Then $X$ is said to be adapted to $\mathcal{F}$ if, for every time $t$, the variable $X_t$ is measurable with respect to $\mathcal{F}_t$ and $\mathcal{E}$. Since $\mathcal{F}$ is increasing, this is equivalent to saying that, for each $t$, the numerical random variables $f \circ X_s$ belong to $\mathcal{F}_t$ for all $f$ in $\mathcal{E}$ and all times $s \leq t$.

Every stochastic process is automatically adapted to the filtration it generates. Thus, if $\mathcal{G}$ is the filtration generated by $X$, saying that $X$ is adapted to $\mathcal{F}$ is the same as saying that $\mathcal{F}$ is finer than $\mathcal{G}$.

**Stopping times**

1.1 **Definition.** Let $\mathcal{F}$ be a filtration on $\mathbb{T}$. A random time $T : \Omega \mapsto \mathbb{\bar{T}} = \mathbb{T} \cup \{+\infty\}$ is called a stopping time of $\mathcal{F}$ if

\[ \{T \leq t\} \in \mathcal{F}_t \quad \text{for each} \quad t \in \mathbb{T}. \]

1.2 **Remarks.** The condition 1.2 is equivalent to requiring that the process

\[ Z_t = 1_{\{T \leq t\}}, \quad t \in \mathbb{T}, \]

be adapted to $\mathcal{F}$. When $\mathbb{T}$ is $\mathbb{N}$ or $\mathbb{N}$, this is equivalent to requiring that

\[ \hat{Z}_n = 1_{\{T = n\}}, \quad n \in \mathbb{N}, \]

be adapted to $(\mathcal{F}_n)$; this follows from the preceding remark by noting that $\hat{Z}_n = Z_n - Z_{n-1}$.

Heuristically, a random time signals the occurrence of some physical event. The process $Z$ defined by 1.4 is indeed the indicator of whether that event has or has not occurred: $Z_t(\omega) = 0$ if $t < T(\omega)$ and $Z_t(\omega) = 1$ if $t \geq T(\omega)$. Recalling the heuristic meaning of adaptedness, we conclude that $T$ is a stopping time of $\mathcal{F}$ if the information flow $\mathcal{F}$ enables us to detect the occurrence of that physical event as soon as it occurs, as opposed to inferring its occurrence sometime later. In still other words, $T$ is a stopping time of $\mathcal{F}$ if the information flow $\mathcal{F}$ is such that we can tell what $T(\omega)$ is at the time $T(\omega)$, rather than by inference at some time after $T(\omega)$. These heuristic remarks are more transparent when the time set is $\mathbb{N}$.

The following mental test incorporates all these remarks into a virtual alarm system. Imagine a computer that is being fed the flow $\mathcal{F}$ of information and that is capable of checking, at each time $t$, whether $\omega \in H$ for every possible $\omega$ in $\Omega$ and every event $H$ in $\mathcal{F}_t$. If it is possible to attach to it an