This chapter will deal briefly with the results stated and proved by Noether in the *Invariante Variationsprobleme*\(^1\) [1918c]. Her originality in this article consisted in dealing with problems that arose either in classical mechanics (the first theorem) or in general relativity (the second theorem). We emphasize what has been ignored by most authors who have cited this article, that in it Noether treated a problem of very great generality, since she dealt with a Lagrangian of arbitrary order with an arbitrary number of independent variables,\(^2\) as well as an arbitrary number of dependent variables, and considered the invariance of such Lagrangians under the action of “groups of infinitesimal transformations.” The infinitesimal transformations in question that form, in modern mathematical terminology, Lie algebras of finite dimension, \(\rho\), or of infinite dimension are genuine generalizations of the usual vector fields since their components depend not only on the independent and dependent variables, as is the case for the infinitesimal generators of Lie groups of transformations, but also on the successive derivatives of the dependent variables. In other words, the infinitesimal symmetries that she considered might depend not only on the field variables but also on their derivatives of order 1 or higher.\(^3\)

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\(^1\) For a more mathematical discussion, see Olver [1986a], as well as the numerous references cited therein, or see Kosmann-Schwarzbach [1985] and [1987].

\(^2\) This theory has since been developed by numerous authors (see, *infra*, Chap. 7). Leonid Dickey [1991] [1994] calls it the “multi-time” theory if there are more than one independent variable. Other authors reserve that description for the case in which several independent variables play a role analogous to that of time among the four variables of relativistic space-time. Recall that variational problems with the time as the single independent variable correspond to problems in mechanics, while those with several independent variables arise in field theory. In nonrelativistic field theory, the three independent variables represent coordinates in space while, in relativistic field theory, the four independent variables represent coordinates in space-time.

\(^3\) Recall that Noether uses the term “invariance” rather than the terms “symmetry” or “symmetry transformation” which have now become standard. She distinguishes between global transformations that form a “continuous group in the sense of Lie” and the infinitesimal transformations that are the generators of the one-parameter subgroups of such a group, while the vanishing of a Lie derivative is expressed by “Lie’s differential equation.” We often abbreviate “infinitesimal symmetry” as “symmetry.”
2.1 Preliminaries

Let us recall that a conservation law in mechanics is a quantity that depends on the configuration variables and their derivatives, and which remains constant during the motion of the system. Therefore, a conservation law is also called a first integral of the equation of motion. In a field theory described by an evolution equation of the form $\frac{\partial u}{\partial t} = F(x, u, u_x, \ldots)$, a conservation law is a relation of the form $\frac{\partial T}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial A_i}{\partial x_i} = 0$, where the $x = (x_1, \ldots, x_{n-1})$ are the space variables and $t = x_n$ is time, and where $A_1, \ldots, A_{n-1}$ and $T$ are functions of the independent variables, and of the field variables $u$ and their derivatives with respect to the space variables, which relation is satisfied when the field equations are satisfied. In physics a conservation law is also called a continuity equation. If the conditions for the vanishing of the quantities being considered at the boundary of a domain of the space variables, $x_1, \ldots, x_{n-1}$, are satisfied, one deduces, by an application of Stokes’s theorem,\(^4\) that the integral of $T$ over this domain is constant over the course of time. One then says that $T$ is the density of a conserved quantity. More generally, in the presence of several variables, when no single one representing time is distinguished from the others, conservation laws define integrals which depend exclusively on the boundary of the domain of integration. In particular, in the case of two independent variables, one obtains line integrals which depend exclusively on the endpoints of the path under consideration.\(^5\)

In the short introduction to her article, Noether cites, in the text or in the notes, the earlier work of Hamel\(^6\) [1904a, b], Herglotz\(^6\) [1911], Lorentz\(^6\) and his student Fokker \([1917]\), Weyl,\(^7\) Klein\(^8\) [1918b] and Kneser \([1918]\).\(^8\) She explains that her work is based on “a combination of the methods of the formal calculus of variations and Lie’s theory of groups”\(^9\) and also that there is a close relation between her work and Klein’s\([1918b]\).

\(^4\) Stokes’s theorem, also called the Gauss–Ostrogradsky theorem or formula, states that the integral of an exact form $\int \beta$ over a domain $\Omega$ is equal to the integral of $\beta$ on the boundary of $\Omega$; in particular, the integral of the divergence of a vector field on a domain is equal to an integral on the boundary of that domain. This result was due to George Gabriel Stokes (1819–1903), and eventually the general formula came to bear his name.

\(^5\) This is the type of conservation law that is to be found in continuum mechanics and particularly in elasticity theory. See, \textit{infra}, p. 147.

\(^6\) Lorentz’s articles \([1915]\) \([1916]\) on Einstein’s theory of gravitation appeared between 1915 and 1917.

\(^7\) She was probably referring to the article that Weyl had submitted to the \textit{Annalen der Physik} on 8 August 1917 (Weyl \([1917]\)), and maybe also to “Zeit, Raum, Materie” (Weyl \([1918b]\)), which Klein had cited in his note \([1918b]\).

\(^8\) See Chap. 1, pp. 35–37 and 39, for comments on the work of Noether’s predecessors that she cites here.

\(^9\) See, \textit{infra}, Chap. 7, p. 138, considerations of modern developments of the first of these two theories. The theory of Lie groups is now a vast domain of pure mathematics and an indispensable tool in modern physics.