Exact Penalty in Constrained Optimization

2.1 Problems with a locally Lipschitzian constraint function

Let \((X, \| \cdot \|)\) be a Banach space and \((X^*, \| \cdot \|_*)\) its dual space. For each \(x \in X\), each \(x^* \in X^*\) and each \(r > 0\) set

\[ B(x, r) = \{ y \in X : \| y - x \| \leq r \} , \quad B^*(x^*, r) = \{ l \in X^* : \| l - x^* \|_* \leq r \} . \]

Let \( f : X \to R^1 \) be a locally Lipschitzian function. For each \( x \in X \) let

\[ f^0(x, h) = \limsup_{t \to 0^+, y \to x} \frac{f(y + th) - f(y)}{t}, \quad h \in X \]

be the Clarke generalized directional derivative of \( f \) at the point \( x \) [21], let

\[ \partial f(x) = \{ l \in X^* : f^0(x, h) \geq l(h) \text{ for all } h \in X \} \]

be Clarke’s generalized gradient of \( f \) at \( x \) [21] and set

\[ \Xi_f(x) = \inf \{ f^0(x, h) : h \in X \text{ and } ||h|| = 1 \} \]

[100].

A point \( x \in X \) is called a critical point of \( f \) if \( 0 \in \partial f(x) \). It is not difficult to see that \( x \in X \) is a critical point of \( f \) if and only if \( \Xi_f(x) \geq 0 \).

A real number \( c \in R^1 \) is called a critical value of \( f \) if there is a critical point \( x \) of \( f \) such that \( f(x) = c \).

It is known that \( \partial(-f)(x) = -\partial f(x) \) for any \( x \in X \) (see Section 2.3 of [21]). This equality implies that \( x \in X \) is a critical point of \( f \) if and only if \( x \) is a critical point of \(-f\) and \( c \in R^1 \) is a critical value of \( f \) if and only if \(-c\) is a critical value of \(-f\).

For each function \( f : X \to R^1 \) set \( \inf(f) = \inf \{ f(z) : z \in X \} \). For each \( x \in X \) and each \( B \subset X \) put
\[ d(x, B) = \inf\{||x - y|| : y \in B\}. \]

Let \( f : X \to \mathbb{R}^1 \) be a function which is Lipschitzian on all bounded subsets of \( X \) and which satisfies the following growth condition:

\[ \lim_{||x|| \to \infty} f(x) = \infty. \quad (2.1) \]

It is easy to see that \( \inf\{f(z) : z \in X\} > -\infty. \)

Let \( g : X \to \mathbb{R}^1 \) be a locally Lipschitzian function which satisfies the following Palais–Smale (P-S) condition \([8, 76, 100]\):

If \( \{x_i\}_{i=1}^\infty \subset X \), the sequence \( \{g(x_i)\}_{i=1}^\infty \) is bounded and if

\[ \lim \inf_{i \to \infty} \Xi g(x_i) \geq 0, \]

then there is a norm convergent subsequence of \( \{x_i\}_{i=1}^\infty \).

Let \( c \in \mathbb{R}^1 \) be such that \( g^{-1}(c) \) is nonempty.

We consider the constrained problems

minimize \( f(x) \) subject to \( x \in g^{-1}(c) \) \( (P_e) \)

and

minimize \( f(x) \) subject to \( x \in g^{-1}((-\infty, c]) \). \( (P_i) \)

We associate with these two problems the corresponding families of unconstrained minimization problems

minimize \( f(x) + \lambda|g(x) - c| \) subject to \( x \in X \) \( (P_{\lambda e}) \)

and

minimize \( f(x) + \lambda \max\{g(x) - c, 0\} \) subject to \( x \in X \) \( (P_{\lambda i}) \)

where \( \lambda > 0. \)

Set

\[ \inf(f; c) = \inf\{f(z) : z \in g^{-1}(c)\}, \quad (2.2) \]

\[ \inf(f; (-\infty, c]) = \inf\{f(z) : z \in X \text{ and } g(z) \leq c\}. \quad (2.3) \]

The next theorem is the main result of this section.

**Theorem 2.1.** Assume that the number \( c \) is not a critical value of the function \( g \). Then there exist numbers \( \lambda_0 > 0 \) and \( \lambda_1 > 0 \) such that for each positive number \( \epsilon \) there exists \( \delta \in (0, \epsilon) \) such that the following assertions hold:

1. For each \( \lambda > \lambda_0 \) and each \( x \in X \) which satisfies

\[ f(x) + \lambda|g(x) - c| \leq \inf\{f(z) + \lambda|g(z) - c| : z \in X\} + \delta \]

there exists \( y \in g^{-1}(c) \) such that

\[ ||y - x|| \leq \epsilon \text{ and } f(y) \leq \inf(f; c) + \lambda_1 \epsilon. \]