Chapter 5
An Algebra of Pieces of Space — Hermann Grassmann to Gian Carlo Rota
Invited Chapter

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Abstract We sketch the outlines of Gian Carlo Rota’s interaction with the ideas that Hermann Grassmann developed in his Ausdehnungslehre[13, 15] of 1844 and 1862, as adapted and explained by Giuseppe Peano in 1888. This leads us past what Gian Carlo variously called Grassmann-Cayley algebra and Peano spaces to the Whitney algebra of a matroid, and finally to a resolution of the question “What, really, was Grassmann’s regressive product?” This final question is the subject of ongoing joint work with Andrea Brini, Francesco Regonati, and William Schmitt.

5.1 Almost Ten Years Later

We are gathered today in order to renew and deepen our recollection of the ways in which our paths intersected that of Gian Carlo Rota. We do this in poignant sadness, but with a bitter-sweet touch: we are pleased to have this opportunity to meet and to discuss his life and work, since we know how Gian Carlo transformed us through his friendship and his love of mathematics.

We will deal only with the most elementary of geometric questions; how to represent pieces of space of various dimensions, in their relation to one another. It’s a simple story, but one that extends over a period of some 160 years. We’ll start and finish with Hermann Grassmann’s project, but the trail will lead us by Giuseppe Peano, Hassler Whitney, to Gian Carlo Rota and his colleagues.

Before I start, let me pause for a moment to recall a late afternoon at the Accademia Nazionale dei Lincei, in 1973, on the eve of another talk I was petrified to give, when Gian Carlo decided to teach me how to talk, so I wouldn’t make a fool of myself the following day. The procedure was for me to start my talk, with an audience of one, and he would interrupt whenever there was a problem. We were in that otherwise empty conference hall for over two hours, and I never got past my first
paragraph. It was terrifying, but it at least got me through the first battle with my fears and apprehensions, disguised as they usually are by timidity, self-effacement, and other forms of apologetic behavior.

5.2 Synthetic Projective Geometry

Grassmann’s plan was to develop a purely formal algebra to model natural (synthetic) operations on geometric objects: flat, or linear pieces of space of all possible dimensions. His approach was to be synthetic, so that the symbols in his algebra would denote geometric objects themselves, not just numbers (typically, coordinates) that could be derived from those objects by measurement. His was not to be an algebra of numerical quantities, but an algebra of pieces of space.

In the analytic approach, so typical in the teaching of Euclidean geometry, we are encouraged to assign “unknown” variables to the coordinates of variable points, to express our hypotheses as equations in those coordinates, and to derive equations that will express our desired conclusions.

The main advantage of a synthetic approach is that the logic of geometric thought and the logic of algebraic manipulations may conceivably remain parallel, and may continue to cast light upon one another. Grassmann expressed this clearly in his introduction to the *Ausdehnungslehre* [13, 14]:

Grassmann (1844): “Each step from one formula to another appears at once as just the symbolic expression of a parallel act of abstract reasoning. The methods formerly used require the introduction of arbitrary coordinates that have nothing to do with the problem and completely obscure the basic idea, leaving the calculation as a mechanical generation of formulas, inaccessible and thus deadening to the intellect. Here, however, where the idea is no longer strangely obscured but radiates through the formulas in complete clarity, the intellect grasps the progressive development of the idea with each formal development of the mathematics”.

In our contemporary setting, a synthetic approach to geometry yields additional benefits. At the completion of a synthetic calculation, there is no need to climb back up from scalars (real numbers, necessarily subject to round-off errors, often rendered totally useless by division by zero) or from drawings, fraught with their own approximations of incidence, to statements of geometric incidence. In the synthetic approach, one even receives precise warnings as to particular positions of degeneracy. The synthetic approach is thus tailor-made for machine computation.

Gian Carlo was a stalwart proponent of the synthetic approach to geometry during the decade of the 1960’s, when he studied the combinatorics of ordered sets and lattices, and in particular, matroid theory. But this attitude did not withstand his encounter with invariant theory, beginning with his lectures on the invariant theory of the symmetric group at the A.M.S. summer school at Bowdoin College in 1971.