Chapter 9
Rota, Probability, Algebra and Logic
Invited Chapter

Daniele Mundici

Abstract Inspired by Rota’s Fubini Lectures, we present the MV-algebraic extensions of various results in probability theory, first proved for boolean algebras by De Finetti, Kolmogorov, Carathéodory, Loomis, Sikorski and others. MV-algebras stand to Łukasiewicz infinite-valued logic as boolean algebras stand to boolean logic. Using Elliott’s classification, the correspondence between countable boolean algebras and commutative AF C*-algebras extends to a correspondence between countable MV-algebras and AF C*-algebras whose Murray-von Neumann order of projections is a lattice. In this way, (faithful, invariant) MV-algebraic states are identified with (faithful, invariant) tracial states of their corresponding AF C*-algebras. Faithful invariant states exist in all finitely presented MV-algebras. At the other extreme, working in the context of σ-complete MV-algebras we present a generalization of Carathéodory boolean algebraic probability theory.

9.1 MV-algebraic States and De Finetti Coherence Criterion

Suppose we are given an arbitrary set \( E = \{X_1, \ldots, X_m\} \) together with a nonempty subset \( W \) of the \( m \)-cube \([0, 1]^{X_1, \ldots, X_m}\). Let us agree to say that \( E \) is a set of “events”, and \( W \) a set of “possible worlds”. Already at this level of extreme generality, we can give a precise definition of \( W \)-coherent probability assessment of \( E \):

Definition 9.1. A map \( \beta: E \to [0, 1] \) is \( W \)-incoherent if for some \( \sigma_1, \ldots, \sigma_m \in \mathbb{R} \) we have \( \sum_{i=1}^{m} \sigma_i(\beta(X_i) - v(X_i)) < 0 \) for all \( v \in W \). Otherwise, \( \beta \) is \( W \)-coherent.

The rationale behind this definition is as follows: Two players, Ada (the bookmaker) and Blaise (the bettor) wager money on the outcome of events \( X_1, \ldots, X_m \) in the set \( W \subseteq [0, 1]^{X_1, \ldots, X_m} \) of possible worlds. Ada proclaims her “degree of belief”
(“betting odd”) \( \beta(X_i) \in [0, 1] \), and Blaise chooses a “stake” \( \sigma_i \in \mathbb{R} \) for his bet on \( X_i \).

After these preliminaries, \( \sigma_i \beta(X_i) \) euro are paid, with the proviso that \( -\sigma_i \beta(X_i) \) euro will be paid back in the possible world \( v \in \mathcal{W} \) assigning the (truth-)value \( v(X_i) \) to each event \( X_i \). Money transfers are oriented so that “positive” means Blaise-to-Ada. In particular, for \( \sigma_i < 0 \), we have a reverse bet: Ada first pays Blaise \( |\sigma_i| \beta(X_i) \) euro, while Blaise will pay back \( |\sigma_i| v(X_i) \) in the possible world \( v \). Ada’s book \( \beta \) would lead her to financial disaster if Blaise could choose stakes \( \sigma_1, \ldots, \sigma_m \) ensuring him to win money in every \( v \in \mathcal{W} \)—or equivalently, to win at least one million euro in every possible world. Such a disastrous book is exactly a \( \mathcal{W} \)-incoherent book in the sense of Definition 9.1.

In the particular case when \( \mathcal{W} \subseteq \{0, 1\}^E \), Definition 9.1 yields De Finetti’s celebrated no-Dutch-Book criterion for coherent probability assessments of yes-no events ([8, §7, p. 308], [9, pp. 6-7], [10, p. 87]).

For any \( E \) and \( \mathcal{W} \), in Theorem 9.1 below we will construct a theory \( \Theta \) in infinite-valued (propositional) Łukasiewicz logic \( \mathcal{L}_\infty \), such that \( \mathcal{W} \)-coherent maps coincide with restrictions to \( E \) of convex combinations of models of \( \Theta \). The necessary preliminary notation and terminology are as follows:

Formulas in \( \mathcal{L}_\infty \) are the same strings of symbols as in boolean logic. While Łukasiewicz [24] used implication \( \to \) as a main connective, together with negation \( \neg \), we will use the connectives \( \odot \) of conjunction and \( \oplus \) of disjunction. We will write \( x \to y \) as an abbreviation of \( \neg x \oplus y \). *Propositional variables* are strings of symbols of the form \( X_1, X_2, \ldots \), and formulas have the usual inductive definition. \( \mathcal{F}_m \) will denote the set of all formulas whose variables are in the set \( \{X_1, \ldots, X_m\} \).

A (Łukasiewicz) valuation of \( \mathcal{F}_m \) is a function \( v : \mathcal{F}_m \to [0, 1] \) such that \( v(\neg \phi) = 1 - v(\phi) \), \( v(\phi \odot \psi) = \max(0, v(\phi) + v(\psi) - 1) \), and \( v(\phi \oplus \psi) = \min(1, v(\phi) + v(\psi)) \), for all \( \phi, \psi \in \mathcal{F}_m \). We say that \( v \) satisfies a set \( \Psi \subseteq \mathcal{F}_m \) if \( v(\theta) = 1 \) for all \( \theta \in \Psi \). The restriction map \( v \mapsto v{|}_{\{X_1, \ldots, X_m\}} \) is a one-one correspondence between valuations of \( \mathcal{F}_m \) and points of the \( m \)-cube \( [0, 1]^{|X_1 \cdots X_m|} \). For any point \( w \) in the \( m \)-cube, we will use the notation

\[
\hat{w} : \mathcal{F}_m \to [0, 1] \text{ for the unique valuation of } \mathcal{F}_m \text{ extending } w. \tag{9.1}
\]

A proper subset \( \Theta \) of \( \mathcal{F}_m \) is said to be a (consistent) theory in the variables \( X_1, \ldots, X_m \) if \( \Theta \) contains every tautology\(^1\) \( \phi \in \mathcal{F}_m \) and is closed under Modus Ponens: whenever \( \psi \) and \( \psi \to \chi \) are in \( \Theta \) then \( \chi \) is in \( \Theta \). We say that a theory \( \Theta \subseteq \mathcal{F}_m \) is finitely axiomatizable if there is a formula \( \theta \in \mathcal{F}_m \) such that \( \Theta \) is the smallest theory in the variables \( X_1, \ldots, X_m \) containing \( \theta \). In this case, following [31, p. 222-223] we write

\[
\Theta = \Theta^+ = \{ \psi \in \mathcal{F}_m \mid \theta \vdash \psi \}, \tag{9.2}
\]

meaning that \( \Theta \) is the set of formulas in the variables \( X_1, \ldots, X_m \) which are consequences\(^2\) of \( \theta \). The set of variables \( X_1, \ldots, X_m \) will always be clear from the context.

\(^1\) As is well known, \( \phi \) is a tautology if \( v(\phi) = 1 \) for every valuation \( v \) of \( \mathcal{F}_m \).

\(^2\) In the present case, Wójtckiecki’s theorem [7, 4.6.7] states that \( \psi \) is a syntactic consequence of \( \theta \) if and only if \( \psi \) is a semantic consequence of \( \theta \) (i.e., \( v(\theta) = 1 \Rightarrow v(\psi) = 1 \) for all valuations \( v \) of \( \mathcal{F}_m \)).