Chapter 14
How do SAT-Solvers Operate?

The purpose of this chapter is to explain how SAT-solvers operate (at least at the
time of the writing of this book, late 2008). Two major families will be described in
this section. The family of SAT-solvers used by the author are based on the Chaff
Algorithm [182], culminating in MINISAT, a popular SAT-solver [104] [6]. This
gives insight into Chapter 13 on Page 245 in particular, by highlighting why the
number of variables per clause, number of clauses, and number of variables, are
taken as the three general barometers of difficulty for a particular SAT problem. As
a contrast, we will first describe Walk-SAT, a very different type of solver, to show
how these methods and approaches differ and are similar. At this time, many SAT-
solvers are in use, most of them of the Chaff/MINISAT type. However, that could
someday change, perhaps even soon.

Besides the Chaff family and the Walk-SAT family, many other SAT algorithms
have been proposed in previous years, and also many preprocessing techniques,
none of which will be described below. While the author makes frequent and ex-
tensive use of SAT-solvers, he is not an expert on their internals. The following is
meant to be informative and general, and so many details are omitted.

14.1 The Problem Itself

Given a logical sentence over certain variables, does there exist a set of assign-
ments of true and false to each of those variables so that the entire sentence eval-
uates as true? This question is the “SAT” problem, and is the root of the theory of
NP-Completeness.

The term “logical sentence” in this book refers to an expression composed of
variables, as well as the operators from predicate calculus (AND, OR, NOT, IM-
PLIES, and IFF), arranged according to the grammar of predicate calculus. There
are no universal quantifiers (i.e. ∀), existential quantifiers (i.e. ∃), or any functions.
An example of such a sentence is
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\((D \land \overline{B} \land A) \Rightarrow (B \lor C)\)

which is satisfied by (for example) setting all the variables to true.

It is a basic fact from digital circuit theory that any logical sentence can be written as a product of sums (Conjunctive Normal Form or CNF) or as a sum of products (Disjunctive Normal Form or DNF). These terms refer to the semiring first introduced in Section 6.2 on Page 81, where addition is logical-OR and multiplication is logical-AND.

Algebraic Normal Form (ANF) is a sum over \(\mathbb{GF}(2)\), or logical-XORs, each input of which is a product (logical-AND). Polynomials over \(\mathbb{GF}(2)\) are thus ANFs, and we desire to make them CNFs, for use with the SAT-solver.

### 14.1.1 Conjunctive Normal Form

A logical sentence in CNF is a set of clauses. Each clause is combined into a large conjunction or AND-gate. Thus the sentence is true if and only if each clause is true. The clauses are themselves OR-gates, or disjunctions. Each variable in the clause can appear negated, or not negated.

Product of Sums or Conjunctive Normal Form has been selected as the universal notation for SAT-solvers for many reasons. One reason is that all predicate calculus sentences can be written in CNF. Another interesting reason is that some sentences can be written with two or fewer variables per clause, and others require three variables at least for some clauses. There does not exist a logical sentence which cannot be written with the restriction of at most three variables per clause. Solving the SAT problem on CNF sentences with at most two variables per clause (2-CNF-SAT) is possible in polynomial time [63, Ch. 34.4]. For CNF sentences with up to three variables per clause (3-CNF-SAT), SAT is NP-Complete. In fact, SAT itself is the “mother problem” of NP-Completeness, in the sense that problems are proven to be NP-Complete by being reduced to SAT.

While one could write any logical sentence in 3-CNF-SAT notation, it is not required for SAT solvers that the author is aware of. The logical sentence need merely be in CNF form.

### 14.2 Solvers like Walk-SAT

In order to explain what the Chaff/Grasp family of SAT-solvers is about, it is worthwhile to investigate a different family, for contrast. The system described here, called Walk-SAT, is by Selman, Kautz and Cohen, and was the leading method for SAT problems for about a decade (very roughly speaking). For certain types of problems, it is still very good. In many ways, Walk-SAT and MINISAT are polar opposites.