Characterization of Special Functions

This chapter is devoted to functional equations connected to a few special functions such as the gamma function, beta function, Riemann’s zeta function, Riemann’s function, Cantor-Lebesgue singular functions, Minkowski’s function, and Dirichlet’s distribution.

14.1 Gamma Function

Euler used the gamma function as an interpolation function for the factorials \( n! = 1 \cdot 2 \cdot \cdots \cdot n \) [261]. Legendre [604] introduced the notation \( \Gamma \) such that \( \Gamma(n) = (n - 1)! \) for \( n \in \mathbb{Z}_+^* \). As a matter of fact, it was D. Bernoulli [121] who gave the first presentation of an interpolating function of the factorials in the form of an infinite product, later known as the gamma function. Euler gave representations by integrals and formulated interesting theorems on the properties of this function. The gamma function appears in physical problems and applications. The gamma function is especially useful to develop other functions that have physical applications. The function

\[
\Gamma(n) = (n - 1)!, \quad \text{for} \ n \in \mathbb{Z}_+^*,
\]

is defined for positive integers \( n \) by recurrence \( \Gamma(1) = 1, \ \Gamma(n + 1) = n\Gamma(n) \).

It is natural to expect to extend this to positive reals as solutions of the equation

\[
f(x + 1) = xf(x) \quad \text{for} \ x > 0.
\]

Some of the well-known representations of the gamma function \( \Gamma \) are (Gronau [324])

\[ (14.3) \quad \Gamma(x) = \lim_{n \to \infty} \frac{n!n^x}{x(x+1) \cdots (x+n)}, \quad x \in ]0, \infty[, \]

\[ (14.4) \quad \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad x > 0, \]

\[ (14.5) \quad \Gamma(x) = \int_0^1 (-\log y)^{x-1}dy, \quad x > 0, \]

\[ (14.6) \quad \Gamma(x) = \int_{-\infty}^\infty e^{sx}e^{-e^s}ds, \quad x > 0, \]

and \( \Gamma \) satisfies, in addition to (14.2) to (14.5),

\[ (14.7) \quad n^{x-\frac{1}{2}} \prod_{k=0}^{n-1} f \left( \frac{x+k}{n} \right) = (2\pi)^{\frac{1}{2}(n-1)} f(x), \quad x > 0, \ n \in \mathbb{Z}_+^*, \]

\[ (14.8) \quad f(x)f(1-x) = \frac{\pi}{\sin \pi x}, \quad x \in ]0, 1[, \]

\[ (14.9) \quad \Gamma(x) = \frac{1}{x} \prod_{n=0}^{\infty} \left( 1 + \frac{1}{n} \right)^x \left( 1 + \frac{x}{n} \right)^{-1}, \quad x > 0 \text{ (due to Euler)}. \]

The integrals (14.4), (14.5), and (14.6) are equivalent. The substitutions \( y = e^{-t}, \ s = \log t \) separately in (14.4) result in (14.5) and (14.6), respectively \( (y = e^s \text{ reduces (14.5) to (14.6)}; t = e^s \text{ transforms (14.6) to (14.4)}) \). The functional equation (14.2) is the most well known among them since it is the simplest, most elegant, and most remarkable formula. It is also known as the \textit{gamma functional equation}.

The logarithmically convex function satisfying (14.2) with \( f(1) = 1 \) is called \textit{Euler's gamma function}.

**Remark.** \( \Gamma(x) \) is not the only solution of (14.2). For example,

\[ f(x) = \Gamma(x)e^{p(x)} \quad \text{or} \quad \Gamma(x)p(x), \quad x > 0, \]

where \( p \) is an arbitrary periodic (analytic) function of period 1 is also a solution of (14.2) (see [554]). Indeed,

\[ f(x + 1) = \Gamma(x + 1)e^{p(x+1)} = x\Gamma(x)e^{p(x)} = xf(x). \]

So, some additional condition is necessary on \( f \) to characterize \( \Gamma \) through (14.2).

**Notation.** For \( n, \nu \in \mathbb{Z}_+ \), define \((\nu)_n = \nu(\nu + 1) \cdots (\nu + n - 1), \ (\nu)_0 = 1. \)

**Theorem 14.1.** (Anastassiadis [76], Bohr and Mollerup [126]). \( \Gamma : ]0, \infty[ \to ]0, \infty[ \) \textit{logarithmically convex is the unique solution of equation (14.2) with} \( \Gamma(1) = 1 \).

**Proof.** Let \( f \) satisfy (14.2) with \( f(1) = 1 \) and be logarithmically convex for \( x > 0. \)

From (14.2) there follows, for \( n \geq 2, \ 0 < x \leq 1, \)

\[ (14.10) \quad f(x + n) = (x)_n f(x), \quad f(n) = (n - 1)!. \]

Since \( f \) is logarithmically convex, \( e^{ax}f(x) \) is also convex for \( x > 0 \) and \( a \) a constant.