Chapter 2
Green’s Functions

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2.1 Introduction

Coupling between mechanical and electric fields has stimulated interesting research related to the microelectromechanical system [1, 2]. The major applications are in sensor and actuator devices by which an electric voltage can induce an elastic deformation and vice versa. Because many novel materials, such as the nitride group semiconductors, are piezoelectric, study on quantum nanostructures is currently a cutting-edge topic with the strain energy band engineering in the center [3, 4]. Novel laminated composites (with adaptive and smart components) are continuously attracting great attention from mechanical, aerospace, and civil engineering branches [5]. In materials property study, the Eshelby-based micromechanics theory has been very popular [6]. In most of these exciting research topics, the fundamental solution of a given system under a unit concentrated force/charge or simply the Green’s function solution is required. This motivates the writing of this chapter. In this chapter, however, only the static case with general anisotropic piezoelectricity is considered, even though a couple of closely related references on vibration and/or dynamics (time-harmonic) wave propagation are briefly reviewed. Furthermore, although emphasis is given to the generalized point and line forces, the Green’s functions to the corresponding point and line dislocations, as well as point and line eigenstrain are also discussed or presented based on Betti’s reciprocal theorem.

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2.2 Governing Equations

Consider a linear, anisotropic piezoelectric and heterogeneous solid occupying the domain $V$ bounded by the boundary $S$. In discussing the Green’s functions, the problem domain and the corresponding boundary conditions are clearly described later. We also assume that the deformation is static, and thus the field equations for such a solid consist of [7]:

(a) *Equilibrium equations (including Gauss equation)*:

\[
\sigma_{ji,j} + f_i = 0 \quad D_{i,i} - q = 0, \quad (2.1)
\]

where $\sigma_{ij}$ and $D_i$ are the stress and electric displacement, respectively; $f_i$ and $q$ are the body force and electric charge, respectively. In this and the following sections, summation from 1 to 3 (1 to 4) over repeated lowercase (uppercase) subscripts is implied. A subscript comma denotes the partial differentiation.

In the Cartesian coordinate system, the equilibrium equations are

\[
\begin{align*}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x &= 0 \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y &= 0 \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z &= 0 \quad (2.2a) \\
\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} - q &= 0. \quad (2.2b)
\end{align*}
\]

In the cylindrical coordinate system, the equilibrium equations are

\[
\begin{align*}
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r &= 0 \\
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + f_\theta &= 0 \quad (2.3a) \\
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z &= 0 \\
\frac{\partial D_r}{\partial r} + \frac{\partial D_\theta}{r \partial \theta} + \frac{\partial D_z}{\partial z} - q &= 0. \quad (2.3b)
\end{align*}
\]

(b) *Constitutive relations*:

\[
\begin{align*}
\sigma_{ij} &= C_{ijlm} \gamma_{lm} - e_{kji} E_k \\
D_i &= \varepsilon_{ijk} \gamma_{jk} + \varepsilon_{ij} E_j, \quad (2.4)
\end{align*}
\]

where $\gamma_{ij}$ is the strain and $E_i$ the electric field; $C_{ijlm}$, $e_{ijk}$, and $\varepsilon_{ij}$ are the elastic moduli, piezoelectric coefficients, and dielectric constants, respectively.

The uncoupled state (purely elastic and purely electric deformation) can be obtained by simply setting $e_{ijk} = 0$. For transversely isotropic piezoelectric materials with the $z$-axis being the material symmetric (or the poling) axis,