The Fourier Transform

5.1 Definition and examples

For a given function $f$ such that $\int_{-\infty}^{\infty} |f(x)| \, dx < \infty$, the Fourier transform of $f$ is defined, for each real number $\omega$, by

$$\mathcal{F} f(\omega) := \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx. \quad (5.1)$$

The idea behind this definition is that, for each value of $\omega$, the value of $\mathcal{F} f(\omega)$ captures the component of $f$ that has the frequency $\omega / (2\pi)$ (and period $2\pi / \omega$).

Example 5.1. The Fourier transform of a Gaussian. Let $f(x) = e^{-Ax^2}$, for some positive constant $A > 0$. Then we have

$$\mathcal{F} f(\omega) = \sqrt{\frac{\pi}{A}} e^{-\frac{\omega^2}{4A}}. \quad (5.2)$$

To prove this, we first need the following fact.

Lemma 5.2. For $A \neq 0$, we have $\int_{-\infty}^{\infty} e^{-Ax^2} \, dx = \sqrt{\frac{\pi}{A}}$.

Proof. Squaring the integral, we get

$$\left( \int_{-\infty}^{\infty} e^{-Ax^2} \, dx \right)^2 = \left( \int_{-\infty}^{\infty} e^{-Ax^2} \, dx \right) \left( \int_{-\infty}^{\infty} e^{-Ay^2} \, dy \right)$$

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-A(x^2+y^2)} \, dx \, dy \]

(polar coordinates) \[ = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-Ar^2} \, r \, dr \, d\theta \]
\[ = \int_{\theta=0}^{2\pi} \left( \lim_{b \to \infty} \frac{1 - e^{-Ab^2}}{2A} \right) \, d\theta \]
\[ = \int_{\theta=0}^{2\pi} \frac{1}{2A} \, d\theta \]
\[ = \frac{\pi}{2A} \text{.} \]

Taking square roots proves the lemma. \[ \square \]

Now to compute the Fourier transform for this example. For each \( \omega \),
\[ \mathcal{F} f(\omega) = \int_{-\infty}^{\infty} e^{-Ax^2} e^{-i\omega x} \, dx \]
\[ = \int_{-\infty}^{\infty} e^{-A(x^2+i\omega x/A)} \, dx \]
\[ \text{(complete the square)} = \int_{-\infty}^{\infty} e^{-A(x^2+i\omega x/A+(i\omega/2A)^2)} e^{A(i\omega/2A)^2} \, dx \]
\[ = e^{-\omega^2/4A} \int_{-\infty}^{\infty} e^{-A(x+i\omega/2A)^2} \, dx \]
\[ = e^{-\omega^2/4A} \int_{-\infty}^{\infty} e^{-Au^2} \, du \text{ with } u = x + i\omega/2A \]
\[ = \sqrt{\frac{\pi}{A}} e^{-\omega^2/4A} \text{ by the lemma.} \]

This establishes the result we were after. \[ \square \]

Observe that if we take \( A = 1/2 \), then \( f(x) = e^{-x^2/2} \) and \( \mathcal{F} f(\omega) = \sqrt{2\pi} e^{-\omega^2/2} \), a constant multiple of \( f \) itself.

Additional examples are considered in the exercises. Let us look at some basic properties of the Fourier transform.

### 5.2 Properties and applications

**Additivity.** Because the integral of a sum of functions is equal to the sum of the integrals of the functions separately, it follows that
\[ \mathcal{F}(f+g)(\omega) = \mathcal{F} f(\omega) + \mathcal{F} g(\omega), \] (5.3)