Analytic $D$-Modules and the de Rham Functor

Although our objectives in this book are algebraic $D$-modules ($D$-modules on smooth algebraic varieties), we have to consider the corresponding analytic $D$-modules ($D$-modules on the underlying complex manifolds with classical topology) in defining their solution (and de Rham) complexes. In this chapter after giving a brief survey of the general theory of analytic $D$-modules which are partially parallel to the theory of algebraic $D$-modules given in earlier chapters we present fundamental properties on the solution and the de Rham complexes. In particular, we give a proof of Kashiwara’s constructibility theorem for analytic holonomic $D$-modules. We note that we also include another shorter proof of this important result in the special case of algebraic holonomic $D$-modules due to Beilinson–Bernstein. Therefore, readers who are interested only in the theory of algebraic $D$-modules can skip reading Sections 4.4 and 4.6 of this chapter.

4.1 Analytic $D$-modules

The aim of this section is to give a brief account of the theory of $D$-modules on complex manifolds. The proofs are occasionally similar to the algebraic cases and are omitted. Readers can refer to the standard textbooks such as Björk [Bj2] and Kashiwara [Kas18] for details.

Let $X$ be a complex manifold. It is regarded as a topological space via the classical topology, and its dimension is denoted by $d_X$. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on $X$, and by $\Theta_X$, $\Omega^p_X$ the sheaves of $\mathcal{O}_X$-modules consisting of holomorphic vector fields and holomorphic differential forms of degree $p$, respectively ($0 \leq p \leq d_X$). We also set $\Omega_X = \Omega^{d_X}_X$. The sheaf $D_X$ of holomorphic differential operators on $X$ is defined as the subring of $\mathcal{E}nd_C(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\Theta_X$. In terms of a local coordinate $\{x_i\}_{1 \leq i \leq n}$ on an open subset $U$ of $X$ we have

$$D_X|_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial^\alpha,$$

where
\[ \partial_i = \frac{\partial}{\partial x_i} \quad (1 \leq i \leq n), \quad \partial^\alpha = \partial^\alpha_1 \cdots \partial^\alpha_n \quad (\alpha = (\alpha_1, \ldots, \alpha_n)) . \]

We have the order filtration \( F = \{ F_l D_X\}_{l \geq 0} \) of \( D_X \) given by

\[ F_l D_X|_U = \sum_{|\alpha| \leq l} \mathcal{O}_U \partial^\alpha \quad (|\alpha| = \sum_i \alpha_i) , \]

where \( U \) and \( \{ x_i \} \) are as above. It satisfies properties parallel to those in Proposition 1.1.3, and \( D_X \) turns out to be a filtered ring. The associated graded ring \( \text{gr} D_X \) is a sheaf of commutative algebras over \( \mathcal{O}_X \), which is canonically regarded as a subalgebra of \( \pi_* \mathcal{O}_{T^* X} \), where \( \pi : T^* X \to X \) denotes the cotangent bundle of \( X \).

Note that we have obvious analogies of the contents of Section 1.2, 1.3. In particular, we have an equivalence

\[ \Omega_X \otimes_{\mathcal{O}_X} (\bullet) : \text{Mod}(D_X) \longrightarrow \text{Mod}(D_X^\text{op}) \]

between the categories \( \text{Mod}(D_X) \), \( \text{Mod}(D_X^\text{op}) \) of left and right \( D_X \)-modules, respectively. Moreover, for a morphism \( f : X \to Y \) of complex manifolds we have a \( (D_X, f^{-1} D_Y) \)-bimodule \( D_X \to Y \) and an \( (f^{-1} D_Y, D_X) \)-bimodule \( D_Y \to X \) of \( \mathcal{O}_X \otimes \mathcal{O}_Y \) \( D_X \to Y \otimes f^{-1} \mathcal{O}_Y \) \( f^{-1} \Omega_Y \otimes \Omega_X \). We say that a \( D_X \)-module is an integrable connection on \( X \) if it is locally free over \( \mathcal{O}_X \) of finite rank.

**Notation 4.1.1.** We denote by \( \text{Conn}(X) \) the category of integrable connections on the complex manifold \( X \).

We have an analogy of Theorem 1.4.10. In particular, \( \text{Conn}(X) \) is an abelian category.

The following result is fundamental in the theory of analytic \( D \)-modules.

**Theorem 4.1.2.**

(i) \( D_X \) is a coherent sheaf of rings.

(ii) For any \( x \in X \) the stalk \( D_{X,x} \) is a noetherian ring with left and right global dimensions \( \dim X \).

The statement (i) follows from the corresponding fact for \( \mathcal{O}_X \) due to Oka, and (ii) is proved similarly to the algebraic case.

We can define the notion of a good filtration on a coherent \( D_X \)-module as in Section 2.1. We remark that in our analytic situation a good filtration on a coherent \( D_X \)-module exists only locally. In fact, there is an example of a coherent \( D_X \)-module which does not admit a global good filtration. Nevertheless, this local existence of a good filtration is sufficient for many purposes. For example, we can define the characteristic variety \( \text{Ch}(M) \) of a coherent \( D_X \)-module \( M \) as follows. For an open subset \( U \) of \( X \) such that \( M|_U \) admits a good filtration \( F \) we have a coherent \( \mathcal{O}_{T^* U} \)-module \( \text{gr}^F(M|_U) := \mathcal{O}_{T^* U} \otimes_{\pi_U^{-1} \text{gr} D_U} \pi_U^{-1} \text{gr}^F M|_U \), where \( \pi_U : T^* U \to U \) denotes the projection. Then the characteristic variety \( \text{Ch}(M) \) is defined to be the closed subvariety of \( T^* X \) such that \( \text{Ch}(M) \cap T^* U = \supp(\text{gr}^F(M|_U)) \) for any \( U \) and \( F \) as above. It is shown to be well defined by Proposition D.1.3.

As in the algebraic case we have the following.