CHAPTER II

Vector Spaces over $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$

Abstract. This chapter introduces vector spaces and linear maps between them, and it goes on to develop certain constructions of new vector spaces out of old, as well as various properties of determinants.

Sections 1–2 define vector spaces, spanning, linear independence, bases, and dimension. The sections make use of row reduction to establish dimension formulas for certain vector spaces associated with matrices. They conclude by stressing methods of calculation that have quietly been developed in proofs.

Section 3 relates matrices and linear maps to each other, first in the case that the linear map carries column vectors to column vectors and then in the general finite-dimensional case. Techniques are developed for working with the matrix of a linear map relative to specified bases and for changing bases. The section concludes with a discussion of isomorphisms of vector spaces.

Sections 4–6 take up constructions of new vector spaces out of old ones, together with corresponding constructions for linear maps. The four constructions of vector spaces in these sections are those of the dual of a vector space, the quotient of two vector spaces, and the direct sum and direct product of two or more vector spaces.

Section 7 introduces determinants of square matrices, together with their calculation and properties. Some of the results that are established are expansion in cofactors, Cramer’s rule, and the value of the determinant of a Vandermonde matrix. It is shown that the determinant function is well defined on any linear map from a finite-dimensional vector space to itself.

Section 8 introduces eigenvectors and eigenvalues for matrices, along with their computation. Also, in this section the characteristic polynomial and the trace of a square matrix are defined, and all these notions are reinterpreted in terms of linear maps.

Section 9 proves the existence of bases for infinite-dimensional vector spaces and discusses the extent to which the material of the first eight sections extends from the finite-dimensional case to be valid in the infinite-dimensional case.

1. Spanning, Linear Independence, and Bases

This chapter develops a theory of rational, real, and complex vector spaces. Many readers will already be familiar with some aspects of this theory, particularly in the case of the vector spaces $\mathbb{Q}^n$, $\mathbb{R}^n$, and $\mathbb{C}^n$ of column vectors, where the tools developed from row reduction allow one to introduce geometric notions and to view geometrically the set of solutions to a set of linear equations. Thus we shall
be brief about many of these matters, concentrating on the algebraic aspects of the theory. Let \( \mathbb{F} \) denote any of \( \mathbb{Q}, \mathbb{R}, \) or \( \mathbb{C} \). Members of \( \mathbb{F} \) are called **scalars**.¹

A **vector space** over \( \mathbb{F} \) is a set \( V \) with two operations, **addition** carrying \( V \times V \) into \( V \) and **scalar multiplication** carrying \( \mathbb{F} \times V \) into \( V \), with the following properties:

(i) the operation of addition, written \( + \), satisfies

(a) \( v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3 \) for all \( v_1, v_2, v_3 \) in \( V \) (associative law),
(b) there exists an element \( 0 \) in \( V \) with \( v + 0 = 0 + v = v \) for all \( v \) in \( V \),
(c) to each \( v \) in \( V \) corresponds an element \( -v \) in \( V \) such that \( v + (-v) = 0 \),
(d) \( v_1 + v_2 = v_2 + v_1 \) for all \( v_1 \) and \( v_2 \) in \( V \) (commutative law);

(ii) the operation of scalar multiplication, written without a sign, satisfies

(a) \( a(bv) = (ab)v \) for all \( v \) in \( V \) and all scalars \( a \) and \( b \),
(b) \( 1v = v \) for all \( v \) in \( V \) and for the scalar \( 1 \);

(iii) the two operations are related by the distributive laws

(a) \( a(v_1 + v_2) = av_1 + av_2 \) for all \( v_1 \) and \( v_2 \) in \( V \) and for all scalars \( a \),
(b) \( (a + b)v = av + bv \) for all \( v \) in \( V \) and all scalars \( a \) and \( b \).

It is immediate from these properties that

- \( 0 \) is unique (since \( 0’ = 0’ + 0 = 0 \)),
- \( -v \) is unique (since \( (-v)’ = (-v)’ + 0 = (-v)’ + (v + (-v)) = ((-v)’ + v) + (-v) = 0 + (-v) = (-v) \)),
- \( 0v = 0 \) (since \( 0v = (0 + 0)v = 0v + 0v \)),
- \( (−1)v = −v \) (since \( 0 = 0v = (1 + (−1))v = 1v + (−1)v = v + (−1)v \)),
- \( a0 = 0 \) (since \( a0 = a(0 + 0) = a0 + a0 \)).

Members of \( V \) are called **vectors**.

**Examples.**

1. \( V = M_{kn}(\mathbb{F}) \), the space of all \( k \)-by-\( n \) matrices. The above properties of a vector space over \( \mathbb{F} \) were already observed in Section I.6. The vector space \( \mathbb{F}^k \) of all \( k \)-dimensional column vectors is the special case \( n = 1 \), and the vector space \( \mathbb{F} \) of scalars is the special case \( k = n = 1 \).

2. Let \( S \) be any nonempty set, and let \( V \) be the set of all functions from \( S \) into \( \mathbb{F} \). Define operations by \( (f + g)(s) = f(s) + g(s) \) and \( (cf)(s) = c(f(s)) \). The operations on the right sides of these equations are those in \( \mathbb{F} \), and the properties of a vector space follow from the fact that they hold in \( \mathbb{F} \) at each \( s \).

¹All the material of this chapter will ultimately be seen to work when \( \mathbb{F} \) is replaced by any “field.” This point will not be important for us at this stage, and we postpone considering it further until Chapter IV.