1 Preliminaries

Let $E$ be a domain in $\mathbb{R}^N$ for some $N \geq 2$, with boundary $\partial E$ of class $C^1$. Points in $E$ are denoted by $x = (x_1, \ldots, x_N)$. A function $u \in C^2(E)$ is harmonic in $E$ if
\[
\Delta u = \text{div} \nabla u = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} u = 0 \quad \text{in} \quad E.
\] (1.1)

The formal operator $\Delta$ is called the Laplacian.\(^1\) The interest in this equation stems from its connection to physical phenomena such as

1. Steady state heat conduction in a homogeneous body with constant heat capacity and constant conductivity.
2. Steady state potential flow of an incompressible fluid in a porous medium with constant permeability.
3. Gravitational potential in $\mathbb{R}^N$ generated by a uniform distribution of masses.

The interest is also of pure mathematical nature in view of the rich structure exhibited by (1.1). The formal operator in (1.1) is invariant under rotations or translations of the coordinate axes. Precisely, if $A$ is a (unitary, orthonormal) rotation matrix and $y = A(x - \xi)$ for some fixed $\xi \in \mathbb{R}^N$, then formally
\[
\Delta_x = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} = \sum_{i=1}^{N} \frac{\partial^2}{\partial y_i^2} = \Delta_y.
\]

This property is also called spherical symmetry of the Laplacian in $\mathbb{R}^N$.

\(^1\)Pierre Simon, Marquis de Laplace, 1749–1827. Author of *Traité de Mécanique Céleste* (1799–1825). Also known for the frequent use of the phrase *il est aisé de voir* which has unfortunately become all too popular in modern mathematical writings. The same equation had been introduced, in the context of potential fluids, by Joseph Louis, Compte de Lagrange, 1736–1813, author of *Traité de Mécanique Analytique* (1788).
1.1 The Dirichlet and Neumann Problems

Given $\varphi \in C(\partial E)$, the Dirichlet problem for the operator $\Delta$ in $E$ consists in finding a function $u \in C^2(E) \cap C(\bar{E})$ satisfying
\[
\Delta u = 0 \text{ in } E, \quad \text{and } u|_{\partial E} = \varphi. \tag{1.2}
\]

Given $\psi \in C(\partial E)$, the Neumann problem consists in finding a function $u \in C^2(E) \cap C^1(\bar{E})$ satisfying
\[
\Delta u = 0 \text{ in } E, \quad \text{and } \frac{\partial}{\partial n} u = \nabla u \cdot n = \psi \text{ on } \partial E \tag{1.3}
\]
where $n$ denotes the outward unit normal to $\partial E$. The Neumann datum $\psi$ is also called variational.

We will prove that if $E$ is bounded, the Dirichlet problem is always uniquely solvable. The Neumann problem, on the other hand, is not always solvable. Indeed, integrating the first of (1.3) in $E$, we arrive at the necessary condition
\[
\int_{\partial E} \psi \, d\sigma = 0 \tag{1.4}
\]
where $d\sigma$ denotes the surface measure on $\partial E$. Thus $\psi$ cannot be assigned arbitrarily.

**Lemma 1.1** Let $E$ be a bounded open set with boundary $\partial E$ of class $C^1$ and assume that (1.2) and (1.3) can both be solved within the class $C^2(\bar{E})$. Then the solution of (1.2) is uniquely determined by $\varphi$, and the solution of (1.3) is uniquely determined by $\psi$ up to a constant.

**Proof** We prove only the statement regarding the Dirichlet problem. If $u_i$ for $i = 1, 2$, are two solutions of (1.2), the difference $w = u_1 - u_2$ is a solution of the Dirichlet problem with homogeneous data
\[
\Delta w = 0 \text{ in } E, \quad w|_{\partial E} = 0.
\]
Multiplying the first of these by $w$ and integrating over $E$ gives
\[
\int_E |\nabla w|^2 \, dx = 0. \quad \blacksquare
\]

**Remark 1.1** Arguments of this kind are referred to as energy methods. The assumption $w \in C^2(\bar{E})$ is used to justify the various calculations in the integration by parts. The lemma continues to hold for solutions in the class $C^2(E) \cap C^1(\bar{E})$. Indeed, one might first carry the integration over an open, proper subset $E' \subset E$, with boundary $\partial E'$ of class $C^1$, and then let $E'$ expand to $E$. We will show later that uniqueness for the Dirichlet problem holds within the class $C^2(E) \cap C(\bar{E})$, required by the formulation (1.2).

**Remark 1.2** A consequence of the lemma is that the problem
\[
\Delta u = 0 \text{ in } E \quad \text{and } u|_{\partial E} = \varphi, \quad \nabla \cdot n = \psi
\]
in general is not solvable.