Integral Equations and Eigenvalue Problems

1 Kernels in $L^2(E)$

Let $E$ be a bounded open set in $\mathbb{R}^N$ with boundary $\partial E$ of class $C^1$. For complex-valued $f$ and $g$ in $L^2(E)$, set

$$\langle f, g \rangle = \int_E f \overline{g} \, dx \quad \text{and} \quad \|f\|^2 = \langle f, f \rangle$$

and say that $f$ and $g$ are orthogonal if $\langle f, g \rangle = 0$. A complex-valued $dx \times dx$-measurable function $K(\cdot; \cdot)$ defined in $E \times E$ is a kernel acting in $L^2(E)$ if the two operators

$$Af = \int_E K(\cdot; y)f(y)dy, \quad A^*f = \int_E \overline{K}(y; \cdot)f(y)dy$$

map bounded subsets of $L^2(E)$ into bounded subsets of $L^2(E)$, equivalently, if there is a constant $\gamma$ depending only upon $N$ and $E$ such that

$$\|Af\| \leq \gamma \|f\| \quad \text{and} \quad \|A^*f\| \leq \gamma \|f\|, \quad \text{for all } f \in L^2(E).$$

It would be sufficient to require only one of these, since any of them implies the other. Indeed, assuming that the first holds

$$\left|\langle A^*f, g \rangle \right| = \left| \int_E \left( \int_E \overline{K}(y; x)f(y)dy \right) g(x)dx \right|$$

$$= \left| \int_E f(y) \left( \int_E K(y; x)g(x)dx \right)dy \right| = |\langle f, Ag \rangle| \leq \gamma \|f\| \|g\|.$$
Their norm is defined by

\[ \|A\| = \sup_{\|f\| = 1} \|Af\|, \quad \|A^*\| = \sup_{\|f\| = 1} \|A^*f\|. \]

By the characterization of the \( L^2(E) \)-norm ([31], Chapter V, Section 4)

\[
\|A\| = \sup_{\|f\| = 1} \sup_{\|g\| = 1} \left| \int_E \left( \int_E K(x;y)f(y)dy \right) \tilde{g}(x)dx \right|
\]

\[
= \sup_{\|f\| = 1} \sup_{\|g\| = 1} \left| \int_E f(y) \left( \int_E \tilde{K}(x;y)g(x)dx \right) dy \right|
\]

\[
\leq \sup_{\|f\| = 1} \sup_{\|g\| = 1} \|f\| \left( \int_E \left( \int_E \tilde{K}(x;y)g(x)dx \right)^2 dy \right)^{1/2}
\]

\[
= \sup_{\|g\| = 1} \left( \int_E \left( \int_E \tilde{K}(x;y)g(x)dx \right)^2 dy \right)^{1/2} = \|A^*\|.
\]

A similar calculation gives \( \|A^*\| \leq \|A\| \). Thus \( \|A\| = \|A^*\| \).

A kernel \( K(\cdot;\cdot) \) in \( L^2(E) \) is compact if the resulting operators \( A \) and \( A^* \) are compact in \( L^2(E) \), i.e., if they map bounded subsets of \( L^2(E) \) into pre-compact subsets of \( L^2(E) \). If \( A \) is compact, \( A^* \) is also compact.

A kernel \( K(\cdot;\cdot) \) in \( L^2(E) \) is symmetric if \( K(x;y) = K(y,x) \) for a.e. \((x,y) \in E \times E \). If it is symmetric and real-valued, then \( A = A^* \).

1.1 Examples of Kernels in \( L^2(E) \)

Given two \( n \)-tuples \( \{\varphi_1, \ldots, \varphi_n\} \) and \( \{\psi_1, \ldots, \psi_n\} \) of linearly independent, complex-valued functions in \( L^2(E) \), set

\[
K(x;y) = \sum_{i=1}^n \varphi_i(x)\bar{\psi}_i(y) \quad \text{for a.e. } (x,y) \in E \times E. \tag{1.1}
\]

A kernel of this kind is called separable, or of finite rank, or degenerate. For such a kernel for any \( f \in L^2(E) \)

\[
\int_E K(\cdot;y)f(y)dy = \sum_{i=1}^n \varphi_i(f,\psi_i).
\]

Thus separable kernels are compact, but need not be symmetric. Green’s function \( G(\cdot;\cdot) \) for the Laplacian with homogeneous Dirichlet data is a real-valued, symmetric, compact kernel in \( L^2(E) \) (see Theorem 8.1 of Chapter 3). This last example shows that a kernel \( K(\cdot;\cdot) \) in \( L^2(E) \) need not be in \( L^2(E \times E) \).