New Results on Weight-Two Motivic Cohomology

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Dedicated to A. Grothendieck on his 60th birthday

Introduction

Among Grothendieck’s manifold contributions to algebraic geometry is his emphasis on the search for a universal cohomology theory for algebraic varieties and a conjectured description of it in terms of motives [Ma]. Various authors have recently set out to describe the properties of and conjecturally define a cohomology theory for algebraic varieties, which has been baptized “motivic cohomology” by Beilinson, MacPherson, and Schechtman ([BMS],[Be],[Bl],[T],[L1],[L2]). It is hoped that this theory, when and if it is fully developed, will in some sense be universal and thus provide at least a partial response to Grothendieck’s question.

Meanwhile, it should be pointed out that to a greater or lesser extent, all of the currently proposed definitions are ad hoc and so ultimately unsatisfactory. Presumably there will one day be a natural definition of motivic cohomology and the present attempts will fall into place as calculating devices which explicitly realize this definition (much as the “bar-construction” definition of group cohomology realizes the more natural derived-functor definition).

Let $X$ be a regular noetherian scheme. Beilinson ([Be]) and the author ([L1]) have proposed the existence for each non-negative integer $n$ of certain complexes of sheaves $\Gamma(n, X)$ on $X$ satisfying certain properties, or “axioms”. These complexes will be referred to as “motivic-cohomology

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complexes" and their hypercohomology as "motivic cohomology". Beilinson considered sheaves for the Zariski topology; the author used the étale site. We recall the étale site "axioms":

1. \( \Gamma(0, X) = \mathbb{Z} \), \( \Gamma(1, X) = G_m[-1] \).
2. For \( r \geq 1 \), \( \Gamma(r, X) \) is acyclic outside of \([1, r]\).
3. Let \( \alpha_* \) be the functor which assigns to every étale sheaf on \( X \) the associated Zariski sheaf. Then the Zariski sheaf \( R^{r+1} \alpha_* \Gamma(q, X) = 0 \).
4. Let \( n \) be a positive integer prime to all residue field characteristics of \( X \). Then there exists a distinguished triangle in the derived category

\[
\Gamma(r, X) \xrightarrow{n} \Gamma(r, X) \rightarrow \mu_n^\otimes r \rightarrow \Gamma(r, X)[1].
\]

5. There are product mappings \( \Gamma(r, X) \otimes \Gamma(s, X) \rightarrow \Gamma(r + s, X) \), satisfying the usual properties.

6. The cohomology sheaves \( \mathcal{H}^i(X, \Gamma(r, X)) \) are isomorphic to the étale sheaves \( \text{gr}^r K_{r-1}^q(X) \) up to torsion involving primes \( \leq (r - 1) \).

In our paper \([L2]\) we constructed a candidate for \( \Gamma(2, X) \). Namely, let \( A \) be a regular noetherian ring. Let \( W = \text{Spec} \ A[T] \), \( Z = \text{Spec} \ A[T]/T(T - 1) \).

Let \( B = \{b_1, b_2, \ldots, b_n\} \) be a finite sequence of "exceptional units" of \( A \), i.e., \( b_i \) and \( 1 - b_i \) are both units for all \( i \). Let \( Y_B = \text{Spec} \ A[T]/\prod_{i=1}^n (T - b_i) \).

Then there is an exact sequence

\[
K_3(A) \rightarrow K_2(W - Y_B, Z) \xrightarrow{\varphi_{A,B}} K_1'(Y_B) \rightarrow K_2(A).
\]

Let \( C_{2,1}(A) = \lim_B K_2(W - Y_B, Z) \), \( C_{2,2}(A) = \lim_B K_1'(Y_B) \) and \( \varphi_A = \lim B \varphi_{A,B} \). Let \( \Gamma(2, A) \) be the two-term complex \( (C_{2,1}(A) \xrightarrow{\varphi_A} C_{2,2}(A)) \) with \( C_{2,1}(A) \) in degree 1 and \( C_{2,2}(A) \) in degree 2.

Now if \( U = \text{Spec} \ A \) is an open affine étale over \( X \), the functors \( U \rightarrow C_{2,i}(A) \) for \( i = 1, 2 \) evidently determine a two-term complex of presheaves on \( X \) for the étale topology and we define \( \Gamma(2, X) \) to be the associated complex of sheaves.

Evidently \( \Gamma(2, X) \) satisfies Axiom 1. In our paper \([L2]\) Axiom 2 was proved up to 2-torsion and \( p \)-torsion if \( X \) was a scheme of finite type over a field of characteristic \( p \geq 0 \). In this paper we prove Axiom 2 up to 2-torsion. In \([L2]\) Axioms 3 and 5 were proven for \( X \) finite type over a field and Axiom 4 for all \( X \). Axiom 6 was proved only for \( X = \text{Spec} \ F \), a field, up to \( p \)-torsion. In this paper we prove Axiom 6 for fields, and prove it in general up to 2-torsion.