Resolution of algebraic equations
by theta constants

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The history of algebraic equations is very long. The necessity and the trial of solving algebraic equations existed already in the ancient civilizations. The Babylonians solved equations of degree 2 around 2000 B.C. as well as the Indians and the Chinese. In the 16th century, the Italians discovered the resolutions of the equations of degree 3 and 4 by radicals known as Cardano's formula and Ferrari's formula. However in 1826, Abel [1] (independently about the same epoch Galois [7]) proved the impossibility of solving general equations of degree \( \geq 5 \) by radicals. This is one of the most remarkable event in the history of algebraic equations. Was there nothing to do in this branch of mathematics after the work of Abel and Galois? Yes, in 1858 Hermite [8] and Kronecker [15] proved that we can solve the algebraic equation of degree 5 by using an elliptic modular function. Since \( \eta_a = \exp((1/n) \log a) \) which is also written as \( \exp((1/n) \int_1^a (1/x) dx) \), to allow only the extractions of radicals is to use only the exponential. Hence under this restriction, as we learn in the Galois theory, we can construct only compositions of cyclic extensions, namely solvable extensions. The idea of Hermite and Kronecker is as follows; if we use another transcendental function than the exponential, we can solve the algebraic equation of degree 5. In fact their result is analogous to the formula \( \eta_a = \exp((1/n) \int_1^a (1/x) dx) \). In the
quintic equation they replace the exponential by an elliptic modular function and the integral $\int (1/x)dx$ by elliptic integrals. Kronecker [15] thought the resolution of the equation of degree 5 by an elliptic modular function would be a special case of a more general theorem which might exist. Kronecker's idea was realized in few cases by Klein [11], [13]. Jordan [10] showed that we can solve any algebraic equation of higher degree by modular functions. Jordan's idea is clarified by Thomae's formula, §8 Chap. III (cf. Lindemann [16]). In this appendix, we show how we can deduce from Thomae's formula the resolution of algebraic equations by a Siegel modular function which is explicitly expressed by theta constants (Theorem 2). Therefore Kronecker's idea is completely realized. Our resolution of higher algebraic equations is also similar to the formula $\frac{b}{a} = \exp((1/n) \int_1^a (1/x)dx)$. In our resolution the exponential is replaced by the Siegel modular function and the integral $\int (1/x)dx$ is replaced by hyperelliptic integrals. The existence of such resolution shows that the theta function is useful not only for non-linear differential equations but also for algebraic equations.

Let us fix some notations. We follow in principle the convention of Chap. III. Let $F(X)$ be a polynomial of odd degree $2g+1$ with coefficients in the complex number field $\mathbb{C}$. We assume that the equation $F(X) = 0$ has only simple roots so that $y^2 = F(X)$ defines a hyperelliptic curve $C$ of genus $g$. Then $C$ is a two sheeted covering of $\mathbb{P}^1$ ramified at the roots of $F(X) = 0$ and at $\infty$. Let $x_1, x_2, \ldots, x_{2g+1}$ be the roots