§7. Frobenius' theta formula

In this section we want to combine Riemann's theta formula (II.6) with the Vanishing Property (6.7) of the last section. An amazing cancellation takes place and we can prove that for hyperelliptic \( \Omega \), \( \theta(z, \Omega) \) satisfies a much simpler identity discovered in essence by Frobenius*. We shall make many applications of Frobenius' formula. The first of these is to make more explicit the link between the analytic and algebraic theory of the Jacobian by evaluating the constants \( c_k \) of Theorem 5.3. The second will be to give explicitly via thetas the solutions of Neumann's dynamical system discussed in §4. Other applications will be given in later sections. Because one of these is to the Theorem characterizing hyperelliptic \( \Omega \) by the Vanishing Property (6.7), we want to derive Frobenius' theta formula using only this Vanishing and no further aspects of the hyperelliptic situation. Therefore, we assume we are working in the following situation:

1. \( B \) = fixed set with \( 2g+2 \) elements
2. \( U \subset B \), a fixed subset with \( g+1 \) elements
3. \( \infty \in B-U \) a fixed element
4. \( T \mapsto \eta_T \) an isomorphism:

\[
\begin{array}{c}
\text{even subsets of } B \\
\text{ (modulo } S \sim CS \text{ )}
\end{array} \xrightarrow{\sim} \frac{1}{2} \mathbb{Z}^{2g}/\mathbb{Z}^{2g}
\]

such that

\begin{align*}
& \text{a) } \eta_{S_1 \cdot S_2} = \eta_{S_1} + \eta_{S_2} \\
& \text{b) } e_2(\eta_{S_1}, \eta_{S_2}) = (-1)^{\#S_1 \cap S_2}
\end{align*}

*\text{\textit{U}ber die constanten Factoren der Thetareihen,} Crelle, 98 (1885); 
See top formula, p. 249, Collected Works, vol. II.
\[ \#(T \cup U) - g - 1 \]
\[ e_x(n_T) = (-1)^{\frac{\#(T \cup U) - g - 1}{2}} \]

5. \( \Omega \in \mathcal{L} \) satisfies \( \vartheta[n_\Omega](0, \Omega) = 0 \) if \( \#T \cup U \neq g + 1 \).

6. We fix \( \eta_i \in \frac{1}{2} z^{2g} \) for all \( i \in B - \infty \) such that \( \eta_i \mod z^{2g} \) equals \( \eta_{(i, \infty)} \), and also let \( \eta_\infty = 0 \). (This choice affects nothing essentially.)

We shall use the notation

\[ \varepsilon_S(k) = +1 \text{ if } k \in S \]
\[ -1 \text{ if } k \notin S \]

for all \( k \in B \), subsets \( S \subseteq B \).

**Theorem 7.1 (Generalized Frobenius’ theta formula).** In the above situation, for all \( z_i \in \mathbb{Q}^g, 1 \leq i \leq 4 \) such that
\[ z_1 + z_2 + z_3 + z_4 = 0, \]
and for all \( a_i \in \mathbb{Q}^{2g}, 1 \leq i \leq 4 \), such that
\[ a_1 + a_2 + a_3 + a_4 = 0, \]
then

\[ (F) \quad \sum_{j \in B} \varepsilon_U(j) \prod_{i=1}^{4} \vartheta[a_i + n_j](z_i) = 0 \]

or equivalently:

\[ (F) \quad \sum_{j \in B} \varepsilon_U(j) \exp(4\pi i n_j \cdot \Omega n_j) \prod_{i=1}^{4} \vartheta(z_i + \Omega n_j^i + n_j^i) = 0. \]