CHAPTER X

Methods of Algebraic Geometry

Abstract. This chapter investigates the objects and mappings of algebraic geometry from a geometric point of view, making use especially of the algebraic tools of Chapter VII and of Sections 7–10 of Chapter VIII. In Sections 1–12, \( k \) denotes a fixed algebraically closed field.

Sections 1–6 establish the definitions and elementary properties of varieties, maps between varieties, and dimension, all over \( k \). Sections 1–3 concern varieties and dimension. Affine algebraic sets, affine varieties, and the Zariski topology on affine space are introduced in Section 1, and projective algebraic sets and projective varieties are introduced in Section 3. Section 2 defines the geometric dimension of an affine algebraic set, relating the notion to Krull dimension and transcendence degree. The actual context of Section 2 is a Noetherian topological space, the Zariski topology on affine space being an example. In such a space every closed subset is the finite union of irreducible closed subsets, and the union can be written in a certain way that makes the decomposition unique. Every nonempty closed set has a meaningful geometric dimension. In affine space the irreducible closed sets are the varieties, and each variety acquires a geometric dimension. The discussion in Section 2 applies in the context of projective space as well, and thus each projective variety acquires a geometric dimension. Moreover, any nonempty open subset of a Noetherian space is Noetherian. A nonempty open subset of an affine variety is called quasi-affine, and a nonempty open subset of a projective variety is called quasiprojective. Each quasi-affine variety or quasiprojective variety has a dimension equal to that of its closure, which is a variety.

Sections 4–6 take up maps between varieties. Section 4 introduces spaces of scalar-valued functions on quasiprojective varieties—rational functions, functions regular at a point, and functions regular on an open set. The section goes on to relate these notions for the different kinds of varieties. Section 5 introduces morphisms, which are a restricted kind of function between varieties. The tools of Sections 4–5 together show that for many purposes all the different kinds of varieties can be treated as quasiprojective varieties. Section 6 introduces rational maps between varieties; these are not everywhere-defined functions, but each can be restricted to an open dense subset on which it is a morphism. Rational maps with dense image correspond to field mappings of the fields of rational functions, with the order of the mappings reversed.

Section 7 concerns singularities at points of varieties, still over the field \( k \). Zariski’s Theorem was stated in Chapter VII for affine varieties and partly proved at that time. In the current context it has a meaning for any point of any quasiprojective variety. The section proves the full theorem, which characterizes singular points in a way that shows they remain singular under isomorphisms of varieties.

Section 8 concerns classification questions over \( k \) for irreducible curves, i.e., quasiprojective varieties of dimension 1. From Section 6 it is known that two irreducible curves are equivalent under rational maps if and only if their fields of rational functions are isomorphic. The main theorem of Section 8 is that each such equivalence class of irreducible curves contains an everywhere nonsingular projective curve, and this curve is unique up to isomorphism of varieties. The points of this curve are parametrized by those discrete valuations of the underlying function field that are defined over \( k \).
Sections 9–12 relate the general theory of Sections 1–6 to the topic of solutions of simultaneous
solutions of polynomial equations, as treated at length in Chapter VIII. Section 9 treats monomial
ideals in $\mathbb{k}[X_1, \ldots, X_n]$, identifying their zero loci concretely and computing their dimension. The
section goes on to introduce the affine Hilbert function of this ideal, which measures the proportion of
polynomials of degree $\leq s$ not in the ideal. In the way that this function is defined, it is a polynomial
for large $s$ called the affine Hilbert polynomial of the ideal. Its degree equals the dimension of the
zero locus of the ideal. Section 10 extends this theory from monomial ideals to all ideals, again
concretely computing the dimension of the zero loci, obtaining an affine Hilbert polynomial, and
showing that its degree equals the dimension of the zero locus of the ideal. Section 11 adapts the
theory to homogeneous ideals and projective algebraic sets by making use of the cone in affine
space over the set in projective space. Section 12 applies the theory of Section 11 to address the
question how the dimension of a projective algebraic set is cut down when the set is intersected with
a projective hypersurface. A consequence of the theory is the result that a homogeneous system of
polynomial equations over an algebraically closed field with more unknowns than equations has a
nonzero solution.

Section 13 is a brief introduction to the theory of schemes, which extends the theory of varieties
by replacing the underlying algebraically closed field by an arbitrary commutative ring with identity.

1. Affine Algebraic Sets and Affine Varieties

We come now to the more geometric side of algebraic geometry. At least initially
this means that we are interested in the set of simultaneous solutions of a system
of polynomial equations in several variables. Because of the Nullstellensatz the
natural starting point for the investigation is the case that the underlying field of
coefficients is algebraically closed.

Accordingly, throughout Sections 1–6 of this chapter, $\mathbb{k}$ will denote an alge-
braically closed field.1 We fix a positive integer $n$ and denote by $A$ the polynomial
ring $A = \mathbb{k}[X_1, \ldots, X_n]$. Typical ideals of $A$ will be denoted by $a, b, \ldots$. We
begin by expanding on some definitions made in Section VIII.2. The set

$$A^n = \{(x_1, \ldots, x_n) \in \mathbb{k}^n\}$$

is called affine $n$-space. Members of $A^n$ are called points in affine $n$-space, and
the functions $P \mapsto x_j(P)$ give the coordinates of the points.

To each subset $S$ of polynomials in $A$, we associate the locus of common
zeros, or zero locus of the members of $S$:

$$V(S) = \{ P \in A^n \mid f(P) = 0 \text{ for all } f \in S \}.$$ 

Any such set $V(S)$ is called an affine algebraic set in $A^n$. If $S$ is a finite set
$\{f_1, \ldots, f_k\}$ of polynomials, we allow ourselves to abbreviate $V(\{f_1, \ldots, f_k\})$

1The exposition in these sections is based in part on Chapters 2, 4, and 6 of Fulton’s book,
Chapter I of Hartshorne’s book, and Chapter I of Volume 1 of Shafarevich’s books.