6 Deformation Analysis of Tau Functions

This chapter is devoted to the analysis of the tau functions that were introduced in the preceding chapter. The central observation is that the Green function for the twisted Dirac operator has a finite-rank derivative with respect to the branch points. The finite-dimensional subspace spanned by the “response functions” that arise in this derivative are flat sections for a “Dirac-compatible” connection. The logarithmic derivative of the tau functions with respect to the branch points are shown to be low-order Fourier coefficients in the local expansions for the response functions. The zero-curvature condition for the connection becomes a system of nonlinear deformation equations. Following Sato, Miwa, and Jimbo we find that the low-order Fourier coefficients of response functions can be expressed in terms of the solutions to the deformation equations. For the scaling limit of the Ising two-point function this translates into an expression for the logarithmic derivative that can be written in terms of Painlevé functions of the third kind, a surprising result that was first obtained by Wu, McCoy, Tracy, and Barouch in [159].

We start this chapter with the existence theory for the appropriate Green function. Next we prove that the derivative of the Green function is of finite rank and characterize the response functions. Then we use the formulas for the tau functions given in the preceding chapter to show that the logarithmic derivative of the tau functions is expressed as the level-1 Fourier coefficient for a response function. We then do the zero-curvature local expansion analysis (à la Sato, Miwa, and Jimbo) to see that these Fourier coefficients have a characterization in terms of solutions of the deformation equations. For the two-point scaling functions we reproduce the WMTB formulas. We also describe how this characterization has been used to verify rotational invariance for the $n$-point functions and the Luther–Peschel formulas for the “short distance” asymptotics of the $n$-point Ising scaling functions.
6.1 Boundary Conditions for \( m - D \) on \( E \)

Suppose that \( a = \{a_1, \ldots, a_n\} \) with \( a_i \in \mathbb{C} \) for \( i = 1, \ldots, n \) and \( a_i \neq a_j \) for \( i \neq j \). We introduce a line bundle \( E \) over \( \mathbb{C} \setminus a \) associated with “Ising transition functions” that was already described in Section 3.4 for the case of just one point \( a = a_1 \).

The bundle \( E \) is constructed so that the smooth sections of \( E \) correspond to smooth “multivalued functions” on \( \mathbb{C} \setminus a \) that change by a multiplier \(-1\) when followed around a complete circuit that encloses a single point \( a_i \). To be more precise, note that there are only a finite number of vectors \( a_i - a_j \) for \( i, j = 1, \ldots, n \). Thus it is possible to choose a vector \( r \in \mathbb{C} \setminus \{0\} \) that is not a multiple of any of these vectors. Then the rays defined by

\[
r_i = \{z : z = a_i + tr, \ t > 0\}
\]

do not intersect. Choose an argument \( \theta_r \) for \( r \) so that \( r = |r|e^{i\theta_r} \) with \( |\theta_r| \leq \pi \) and let \( \theta(z) \) denote the polar angle defined by

\[
z = |z|e^{i\theta(z)} \quad \text{with} \quad \theta_r - \pi < \theta(z) < \theta_r + \pi.
\]

This angle is branched along the ray \(-r\). For \( \epsilon > 0 \) define a tubular neighborhood \( t_i \) of \( r_i \) by

\[
t_i(\epsilon) := \left\{ z \in \mathbb{C} \setminus a : \text{dist}(z, r_i) < \epsilon \right\} \cap \left\{ z : |\theta(z - a_i) - \theta_r| < \frac{\pi}{4} \right\}
\]

where \( \text{dist}(z, r_i) \) is the distance from the point \( z \) to the ray \( r_i \). Now choose such an \( \epsilon > 0 \) that the tubular neighborhoods \( t_i \) are mutually disjoint and (for later convenience) such that the disks

\[
D_i(2\epsilon) = \{z : |z - a_i| < 2\epsilon\}
\]

are also mutually disjoint. The situation is pictured in Figure 6.1.

Now we introduce a covering of \( \mathbb{C} \setminus a \) over each element of which the bundle \( E \) is trivial. Let

\[
\mathcal{O}_0 := \{z \in \mathbb{C} \setminus a : z \notin r_i, i = 1, \ldots, n\}
\]

and

\[
\mathcal{O}_i = t_i(\epsilon) \quad \text{for} \ i = 1, \ldots, n.
\]

To define the bundle \( E \) we glue together the trivial bundles

\[
\mathcal{O}_i \times \mathbb{C} \to \mathcal{O}_i
\]

by giving the transition functions \( s_i \) that relate the trivializations over \( \mathcal{O}_0 \) and \( \mathcal{O}_i \). Define

\[
s_i(z) = \begin{cases} 
-1 & \text{for } \theta(z - a_i) < 0, \\
1 & \text{for } \theta(z - a_i) > 0.
\end{cases}
\]