The Polygonal Distribution

Dimitris Karlis\(^1\) and Evdokia Xekalaki\(^2\)

\(^1\) Department of Statistics, Athens University of Economics and Business
\(^2\) Department of Statistics, Athens University of Economics and Business
Department of Statistics, University of California, Berkeley

Abstract: The triangular distribution, although simpler than the beta distribution both for mathematical treatment and for natural interpretation, has not been widely used in the literature as a modelling tool. Applications of this distribution as an alternative to the beta distribution appear to be limited in financial contexts and specifically in the assessment of risk and uncertainty and in modelling prices associated with trading single securities. One of the basic reasons is that it can have only a few shapes. In this paper, a new class of distributions stemming from finite mixtures of the triangular distribution is introduced. Their polygonal shape makes them appealing for modelling purposes since they can be used as simple approximations to several distribution functions. Properties of these distributions are studied and parameter estimation is discussed. Further, the distributions arising when using the triangular distribution instead of the beta distribution as the mixing distribution in the case of two well-known beta mixtures, the beta-binomial and the beta-negative binomial distribution, are examined.

Keywords and phrases: Triangular distribution, binomial mixtures, negative binomial mixtures, triangular-binomial distribution

2.1 Introduction

The probability density function (pdf) of the triangular distribution is given by

\[
f(x \mid \theta) = \begin{cases} 
\frac{2x}{\theta}, & 0 \leq x \leq \theta \\
\frac{2(1-x)}{1-\theta}, & \theta \leq x \leq 1 \\
0, & \text{elsewhere}
\end{cases}
\]  

(2.1)

with “0/0” interpreted as 1. The above definition restricts the random variable \(X\) in the interval \([0, 1]\). One can define in a similar manner triangular distributions in a finite
interval \([\alpha, \beta]\) by considering the transformation \(Y = \frac{X - \alpha}{\beta - \alpha}\). From (2.1), one can see that the density is linearly increasing in the interval \([0, \theta]\) and linearly decreasing in the interval \([\theta, 1]\) (\(\theta\) is the mode of the distribution). The distribution is not symmetric except for the case \(\theta = 1/2\). The parameter \(\theta\) is allowed to take the values 0 and 1, using the appropriate part of the definition given in (2.1). More details about the triangular distribution can be found in van Dorp and Kotz (2004) and the references therein. Johnson (1997) and Johnson and Kotz (1999) refocused interest in the triangular distribution, which appeared to have been ignored as a modeling tool over the last decades, one of the most probable basic reasons being that it can have only a few shapes.

In this paper, a new class of distributions is introduced stemming from finite mixtures of the triangular distribution. Contrary to the triangular distribution, the members of this class have a shape flexibility that makes them appealing for modeling purposes. Because of their shape, which is polygonal, these distributions are termed in the sequel polygonal distributions.

The paper is organized as follows. Following a brief presentation of the triangular distribution in Section 2.2, the polygonal distribution is defined as a finite mixture of triangular component distributions in Section 2.3. Properties of it and estimation are discussed. In Section 2.4, mixture distributions arising when using the triangular as an approximation to a beta mixing distribution are examined. In particular, the cases of beta mixtures of binomial and negative binomial distributions are considered. The paper concludes with some remarks in Section 2.5.

2.2 The Triangular Distribution

We briefly review some properties of the triangular distribution that can have potential use in the context of polygonal distributions.

The triangular distribution consists of two parts that are truncated forms of the \(Beta(\alpha, \beta)\) distribution with density

\[
f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 \leq x \leq 1, \quad \alpha, \beta > 0, B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}. \tag{2.2}
\]

In particular, the first part is a tail truncated \(Beta(2, 1)\) distribution, while the second part a head truncated \(Beta(1, 2)\) distribution, both truncated at \(\theta\). The triangular distribution also arises as the distribution of the mean of two uniform random variables.

The \(s-th\) simple moment of the distribution is given by

\[
\mu_s = \frac{2(1 - \theta^{s+1})}{(s + 1)(s + 2)(1 - \theta)}. \tag{2.3}
\]

The above expression holds for not necessarily integer values of \(s\), which enables computation of non-integral moments of a triangular variate.

Noting that \(1 - \theta^{s+1} = (1 - \theta) \sum_{i=0}^{s} \theta^i\), for integer \(s \geq 0\), simple moments can be rewritten as