Moduli Spaces of $\mathcal{A}$-Connections of Yang–Mills Fields

“Our purpose in this chapter is to expose, in the abstract language that we employ throughout this treatise, the fundamentals of the classical theory indicated by the subject in the title. Roughly speaking, we want to put into perspective the classical and physically, yet mathematically, important (!) theme of the so-called geometry of Yang–Mills equations. This was first advocated by I.M. Singer [1] (see also, for instance, M.F. Atiyah [1: p. 2]). Equivalently, one considers the corresponding space of solutions of the said equations, thus, by definition (see Chapt. I, Definitions 4.1 and 4.2), the space of the Yang–Mills $\mathcal{A}$-connections. However, in view of the physical significance of the “gauge invariant ($\mathcal{A}$-)connections” (see Atiyah’s phrasing in the epigraph above), the same space is finally divided out by the corresponding “gauge group,” so that it is, in effect, the resulting quotient space (“moduli space,” or even “orbit space”) that is under consideration.

Our treatment of this material in this chapter is in accordance with our general goal—to do physics in terms of the standard differential geometry of smooth manifolds by means of the “abstract (“modern”) differential geometry,” since the latter aspect has been started and further exposed in A. Mallios [VS: Vols. I and II]. In this connection, the standard work of Singer is still made in a differential geometric manner, within the context of (smooth) fiber bundle theory.

On the other hand, in Section 7 of Chapter 3 we further consider, according to the standard patterns, (global) gauge fixing (Gribov’s ambiguity) by still following the classical account thereof of I.M. Singer (loc. cit.): Indeed, Singer’s pioneering work was also the main motivation for all the subsequent discussion. Our primary objective here is to put into an abstract perspective the relevant classical material.

1 Preliminaries: The Group of Gauge Transformations or Group of Internal Symmetries

As the title of this section indicates, we consider here the group of transformations. The same group has been discussed in Chapter I, Section 5.1, to consider (Chapt. I, Section 5.2) the “gauge invariance” of the Yang–Mills functional, as well as in

Chapter II, Section 9 of Volume I in connection with the concept of a “local gauge” of a given vector sheaf (loc. cit. (9.8), (9.9) and (9.11)). In this context, see also [VS: Chapt. VI, Section 17], where the same theme is examined.

So, to start with, and in accordance with our general point of view—that is, impose the least possible preassumptions on the particular framework employed—suppose that

we are given a $\mathbb{C}$-algebraized space,

$$\text{(1.1.1)} \quad (X, A),$$

while we also let $E$ be a vector sheaf on $X$, such that

$$\text{(1.1.2)} \quad \text{rk}_A E \equiv \text{rk} E = n \in \mathbb{N}.$$ 

Now, in this context, we first recall that, by definition (see Chapt. I, (5.19)),

the group of gauge transformations of $E$ is given as follows:

$$\text{(1.2)} \quad Aut_A(E) \equiv Aut \equiv (\text{Aut} E)(X) := (\text{End} E)^\ast(X)$$

$$\quad = (\text{End} E)(X)^\ast \equiv (\text{End} E)^\ast.$$ 

In other words, and according to our previous convention, as in (1.2.1), the group under discussion,

$$\text{(1.3)} \quad Aut_A(E) \equiv Aut E,$$

is that one of the $A$-automorphisms of $E$ ($A$-isomorphisms of the given vector sheaf $E$ onto itself). So, by taking the very definition of a sheaf morphism into account (see [VS: Chapt. I, p. 11, Proposition 2.1]), one gets an equivalent and, occasionally, more convenient expression of the latter notion, through a (uniquely defined) morphism of the corresponding (complete) presheaves of (continuous local) sections of the sheaves concerned (loc. cit., Chapt. I, p. 75, (13.19)). Thus, by looking at a particular element (e.g., “gauge transformation” of $E$), say,

$$\text{(1.4)} \quad \phi \in \text{Aut} E = (\text{Aut} E)(X),$$

that is, by definition (see (1.3)), an $A$-automorphism of the given vector sheaf $E$, one obtains, equivalently, a map (i.e., a morphism of (complete) presheaves, as explained above),

$$\text{(1.5)} \quad \phi = (\phi_U),$$

in such a manner that one defines

$$\text{(1.6)} \quad \phi_U \equiv \phi|_U \in \text{Isom}_A(E, E)(U) = \text{Isom}_{A|_U} (E|_U, E|_U)$$

$$\quad \equiv \text{Aut}_{A|_U} (E|_U) \equiv \text{Aut}(E|_U) = (\text{Aut} E)(U).$$