Wave Trace and Poisson Formula

On a compact manifold, the wave trace is defined to be the distributional trace of the wave operator, \( U(t) := e^{it\sqrt{\Delta}} \). This is a spectral invariant because we could write it directly as

\[
\text{tr } U(t) = \sum_{\lambda_j \in \sigma_d(\Delta)} e^{it\sqrt{\lambda_j}}.
\]

(The sum doesn’t converge, but is well-defined as a distribution.) In the early 1970s, work of Colin de Verdière [48, 49], Chazarain [41], and Duistermaat–Guillemin [53], showed that the wave trace has singularities only at values of \( t \) equal to lengths of closed geodesics. Such a connection had previously appeared in the physics literature, for example in the work of Balian–Bloch. The Selberg trace formula itself furnishes a prototype for these results. For compact hyperbolic surfaces one can use it to express the wave trace explicitly as a sum over the length spectrum.

For an infinite-area hyperbolic surface, the trace of the wave operator does not exist even in the distributional sense, but we can define a regularized wave trace using the 0-integral. We will prove in Theorem 11.2 an analogue of the Selberg trace formula expressing this as a sum over the length spectrum. It is far from obvious that the regularized wave trace could also be expressed as a sum over the resonance set \( \mathcal{R}_X \), but fortunately this does turn out to be the case. This is the content of the Poisson formula, Theorem 11.3.

We can illustrate the Poisson formula explicitly in the case of a hyperbolic cylinder, \( C_\ell = \Gamma_\ell \backslash \mathbb{H} \), for which

\[
\mathcal{R}_{C_\ell} = \frac{2\pi i}{\ell} \mathbb{Z} - \mathbb{N}_0 \quad \text{(multiplicity 2)}.
\]

Given a test function, \( \phi \in C_0^\infty(\mathbb{R}_+) \), we observe that

\[
\frac{1}{2} \sum_{\zeta \in \mathcal{R}_{C_\ell}} \int_0^\infty e^{i(\zeta-1/2)t} \phi(t) \, dt = \sum_{n=0}^\infty \sum_{k \in \mathbb{Z}} \int_0^\infty \exp \left( \frac{2\pi ik t}{\ell} - (n + \frac{1}{2})t \right) \phi(t) \, dt
\]
\[
\sum_{k \in \mathbb{Z}} \int_0^\infty e^{2\pi i k t/\ell} \frac{\phi(t)}{2 \sinh t} \, dt = \sum_{k \in \mathbb{Z}} \hat{f}(2\pi k/\ell),
\]

where \( f(t) := \phi(t)/(2 \sinh t/2) \). Noting that \( f \in C_0^\infty(\mathbb{R}) \), we can apply the classical Poisson summation formula (Theorem A.6) to obtain

\[
\sum_{k \in \mathbb{Z}} \hat{f}(2\pi k/\ell) = \ell \sum_{m \in \mathbb{Z}} f(m\ell).
\]

We have thus shown that

\[
\frac{1}{2} \sum_{\zeta \in \mathfrak{R}_{c_\ell}} \int_0^\infty e^{(\zeta-1/2)t} \phi(t) \, dt = \sum_{m \in \mathbb{N}} \frac{\ell}{2 \sinh(m\ell/2)} \delta(t - m\ell).
\]

We could write this more cleanly as a relation between distributions on \( \mathbb{R}_+ \),

\[
\frac{1}{2} \sum_{\zeta \in \mathfrak{R}_{c_\ell}} e^{(\zeta-1/2)t} = \sum_{m=1}^\infty \frac{\ell}{2 \sinh(m\ell/2)} \delta(t - m\ell).
\]

The right-hand side turns out to be precisely the regularized wave trace for the hyperbolic cylinder.

### 11.1 Regularized wave trace

Consider the wave equation on a hyperbolic surface \( X \),

\[
(\partial^2_t + \Delta_X - \frac{1}{4}) u = 0,
\]

with initial conditions \( u|_{t=0} = f \), \( \partial_t u|_{t=0} = g \). The functional calculus allows us to express the general solution as

\[
u = \partial_t W_X(t) f + W(t) g,
\]

using the wave operator

\[
W_X(t) := \sin \left( t \sqrt{\Delta_X - \frac{1}{4}} \right)
\]

and its derivative

\[
\partial_t W_X(t) := \cos \left( t \sqrt{\Delta_X - \frac{1}{4}} \right).
\]

Even in the compact case, the wave trace, \( \text{tr} \partial_t W_X(t) \), is not well-defined as a function, but rather must be interpreted as a distribution on \( \mathbb{R} \). In our infinite-area setting, the trace does not exist even in this sense, but Guillopé–Zworski [87] introduced the following substitute: