Inverse Spectral Geometry

The problem of determining how the spectrum depends on basic properties of a system (e.g., conditions on the metric or potential) is referred to as the “forward” spectral problem. The “inverse” problem is to deduce properties of the original system from some knowledge of the spectrum (including the resonances in the case of a scattering problem). In other words, the goal of inverse spectral theory is to determine exactly how much information about the system is contained in the spectrum. There is a very strong physical motivation for the inverse problem, since the spectrum is often the aspect of a system most accessible to experimental observation. But the question is also natural from a mathematical point of view—the resonances of a hyperbolic surface provide a set of geometric invariants, and the mathematical goal is always to understand the content of such invariants.

**Definition.** Two hyperbolic surfaces are *isospectral* if their resonances sets coincide with multiplicities, and *length isospectral* if their length spectra coincide with multiplicities.

The term “isopolar” is also used in the literature to describe two systems with the same set of scattering poles.

It is known that the resonance set does not determine a hyperbolic surface completely. The first examples of isospectral infinite-area surfaces were pointed out Guillopé–Zworski in [87, Remark 2.3], based on the transplantation method of Bérard [18]. In [29], Brooks–Davidovich applied the Sunada method to construct such examples (see [31] and [30] for details of this approach). Among the possibilities are:

1. Two isospectral hyperbolic surfaces of genus 0 with eight funnels.
2. Two isospectral hyperbolic surfaces of genus 2 with four funnels.
3. Two isospectral hyperbolic surfaces of genus 3 with three funnels.
4. Two isospectral hyperbolic surfaces of genus 0 with sixteen funnels.
5. Families of size $c_1 k c_2 \log k$ of mutually isospectral hyperbolic surfaces of genus $c_3 k$ with $c_4 k$ funnels.
All of these examples are both isospectral and length isospectral. In fact, Sunada
methods produce examples satisfying the stronger condition of having the same rel-
ative scattering phase. Such families are called “isophasal” or “isoscattering.”

These examples are “negative” results showing that the resonance set does not
provide sufficient information to recover the surface. In this chapter we’ll present a
“positive” result limiting the size of isospectral families. First we’ll show that the
resonance set and length spectrum determine each other and fix the topological type,
up to finitely many possibilities. Then we’ll use this information to prove that the
number of hyperbolic surfaces with the same resonance set or length spectrum is
finite.

13.1 Resonances and the length spectrum

An immediate implication of Theorem 10.1 is the analogue of Huber’s theorem
from the compact case, the equivalence of the resonance set and length spectrum.
This was first observed explicitly in Borthwick–Judge–Perry [24]. Of course, such a
connection is also evident from (12.1), which was proven earlier by Guillopé–
Zworski [88].

**Theorem 13.1 (Borthwick–Judge–Perry).** For geometrically finite hyperbolic sur-
face $X$ of infinite area, the resonance set $\mathcal{R}_X$ determines the length spectrum $\mathcal{L}_X$, the
Euler characteristic $\chi(X)$, and the number of cusps $n_c$. The length spectrum deter-
mines $\chi(X)$ and $n_c$ up to a finite number of possibilities. And the length spectrum,
$\chi(X)$, and $n_c$ together determine the resonance set.

**Proof.** Suppose that the resonance set is fixed, which determines $P_X(s)$. We claim
that $P_X(s)$ determines $\chi(X)$, $n_c$, and $q(s)$, in the notation of Theorem 10.1, and
therefore fixes $Z_X(s)$. To see this we take the log of (10.3),

$$\log Z_X(s) = q(s) - \chi(X) \log G_\infty(s) + n_c \log \Gamma(s - \frac{1}{2}) + \log P_X(s),$$

and analyze the asymptotics as $\text{Re } s \to \infty$. Because the sum

$$\log Z(s) = \sum_{\ell \in \Lambda} \sum_{k=0}^{\infty} \log \left(1 - e^{-(s+k)\ell}\right)$$

converges uniformly for $\text{Re } s \geq 1$, it is clear that $\log Z_X(s)$ decays exponentially as
$\text{Re } s \to \infty$. The analogue of Stirling’s formula for the Barnes $G$-function was given
by Voros [215]: for $\text{Re } z > 0$,

$$\log G(z+1) \sim z^2 \left(\frac{1}{2} \log z - \frac{3}{4}\right) + \frac{1}{2} z \log 2\pi - \frac{1}{12} \log z + \zeta'(-1).$$

Together with Stirling’s formula (7.41), this gives an asymptotic formula for $G_\infty$,

$$\log G_\infty(s) \sim -\left(\frac{1}{2}s(s-1) - \frac{1}{6}\right) \log s(s-1) + \frac{3}{2}s(s-1) + 1 + \frac{1}{2} \log 2\pi - 2\zeta'(-1).$$